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Substitution Delone sets with pure point spectrum are inter-model sets

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Abstract

The paper establishes an equivalence between pure point diffraction and certain types of model sets, called inter-model sets, in the context of substitution point sets and substitution tilings. The key ingredients are a new type of coincidence condition in substitution point sets, which we call algebraic coincidence, and the use of a recent characterization of model sets through dynamical systems associated with the point sets or tilings.

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1. Introduction

In the study of aperiodic order, there has been considerable interest in understanding the structure of point sets which have pure point diffraction spectrum. The pure point peaks in the diffraction spectrum are indicative of a highly ordered structure of the point sets, and the pure point diffraction spectrum has provided a new viewpoint for looking at ordered structures. Independent of the diffraction spectrum, there is the spectral theory of dynamical system that arises as the completion of the translational orbit of a point set in the standard Radin–Wolff-type topology [28]. It turns out that pure point dynamical and diffraction spectra are equivalent in quite a general setting (see [22,12,2]). In particular, in substitution point sets, the two types of pure point spectra are equivalent, though not equal usually.

There is a large class of point sets which come from cutting and projecting a lattice in a "higher dimensional" space $\mathbb{R}^d \times H$ into the two lower dimension spaces $\mathbb{R}^d$ and H, where H is a locally compact Abelian group. Discrete point sets in $\mathbb{R}^d$ are obtained by restricting the projection, which maps from $\mathbb{R}^d \times H$ to $\mathbb{R}^d$, to some part of a lattice lying in a cylinder of the form $\mathbb{R}^d \times W$, where the window $W \subset H$ has non-empty interior and compact closure. If a discrete point set in $\mathbb{R}^d$ comes from this projection and its window has the boundary of measure zero, we call it a regular model set (see Section 2 for precise definitions). This has provided a general way of obtaining point sets which have the property of pure point dynamical spectrum (see [14,29,6,20,23]). The inversion problem, that is, determining the

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structure of a discrete set knowing that it is pure point diffractive, is in general impossible to solve. However, with added ingredients it is possible to infer information about the nature of a set from pure point diffractivity. One such piece of information that we can now determine, and that is the aim of this paper to prove, is the structural type of the diffracting set — namely that it is a model set, as long as we have the added ingredient of a primitive substitution.

There is a recent characterization of model sets through the use of dynamical systems associated with point sets in [3] and with multi-colour point sets in [21]. The notion of inter-model sets is introduced in [21] (under the name of model sets) and [3] as a model set satisfying a topological condition which is less restrictive than the boundary condition of a regular model set. In this paper we consider the inter-model sets (see Definition 2.1) and show the equivalence between inter-model sets and pure point dynamical spectrum in the context of primitive substitution point sets.

The main ingredient that establishes the connection between inter-model set and pure point dynamical spectrum is algebraic coincidence. In the literature there are many types of coincidences for substitution point sets and tilings which are equivalent to the property of pure point dynamical spectrum of these sets (see [10,7,31,23,8]). For example, in the class of constant length symbolic substitutions Dekking's coincidence condition is well known. It says the following: suppose that $A = \{a_1, \ldots, a_m\}$ is a finite alphabet with associated set of words A^*, and we are given a primitive substitution $\sigma : A \to A^*$ for which the length l of each word $\sigma(a_i)$ is same, so called "constant length substitution", and the height is 1. Then the associated substitution dynamical system has pure point spectrum if and only if it admits a coincidence, in the sense that there is $k \in \mathbb{Z}_+$ such that $k \le l^n$ for some n and the k-th letter of each word $\sigma^n(a_i)$, $i \le m$, is same. Although the various types of the coincidences are defined in slightly different ways, they fulfill a similar property for pure point dynamical spectrum. In the case of substitution point sets on lattices it is established through *modular* coincidence that regular model set is necessary and sufficient for pure point dynamical spectrum and it is shown that the modular coincidence is computable [23,11]. However there are many examples of substitution point sets and tilings whose underlying structures are not on lattices. For general substitution point sets we introduce in this paper a new type of coincidence called *algebraic coincidence*, whose name comes from the algebraic structure of the point sets, which makes the construction of locally compact Abelian groups and cut and project schemes possible. The Fibonacci substitution sequence is a well-known example of this kind.

Although we are primarily interested in and dealing with Delone multi-colour sets, we need to introduce tilings along the way. It is often advantageous to work with tilings in getting spatial properties of point sets. However it is not necessarily true that a substitution point set can be represented by a tiling in such a way that every point of one type point set is represented by a tile of the same type. In [18], Lagarias and Wang have given a sufficient condition for a substitution point set to be represented by a substitution tiling while maintaining its iteration rules. They call it a self-replicating Delone set. In [23] it is shown that a repetitive substitution point set can be represented by a substitution tiling when a concept called legality of clusters applies. We will talk more about this passage from point sets to tilings in Section 2.3. We make use of this connection in order to derive properties that we need in substitution point sets from substitution tilings.

The main theorem in this paper states (see Theorem 5.3) :

Theorem. *Let* $\mathbf{\Lambda}$ *be a primitive substitution Delone multi-colour set in* $\mathbb{R}^d$ *such that every* $\mathbf{\Lambda}$*-cluster is legal and* $\mathbf{\Lambda}$ *has finite local complexity. Then the following are equivalent:*

(1) $\mathbf{\Lambda}$ *has pure point dynamical spectrum;*
(2) $\mathbf{\Lambda}$ *admits an algebraic coincidence;*
(3) $\mathbf{\Lambda}$ *is an inter-model multi-colour set.*

There are several recent results in the area that we use as new ingredients for the proof of the main theorem. We briefly explain them here.

We can construct two dynamical hulls generated by a point set, using two different topologies. One is a local topology, which defines the closeness of point sets by agreement on large regions around the origin up to small shifts, and the other is a global topology, called the autocorrelation topology, which defines the closeness of point sets looking at how much they agree density-wise up to small shifts. In [3] conditions under which there exists a continuous mapping between the two dynamical hulls are given. In [21], it is shown that this mapping derives inter-model sets from model sets which are projected from open windows.

A Delone set Λ is called a Meyer set if $\Lambda - \Lambda$ is uniformly discrete. The Meyer property is used significantly in the main theorem to show the equivalence between overlap coincidence on substitution tilings and algebraic coincidence on substitution point sets, as well as in [3] and [21] which call upon. However, in [24] it is shown that any substitution point set with pure point dynamical spectrum necessarily has the Meyer property, so we do not need to assume it additionally in the main theorem.

We consider two topologies on the group L generated by the translation vectors of a substitution point set, thereby constructing two locally compact Abelian groups which may be used for defining CPSs. We call one Q-topology and the other P_ϵ-topology. In general, these two topologies on substitution point sets are different. But under the assumption of pure point spectrum, they are equivalent. Since model sets are always associated with their CPSs, it is important to notice from which CPS a model set arises. The equivalence of the two topologies gives us the same CPS, and this allows us to freely use the related results in [3,21].

The proof of the theorem is spread over several sections. The structure of it is as follows: On substitution tilings it is known that overlap coincidence is a necessary and sufficient condition for pure point dynamical spectrum [31]. We introduce algebraic coincidence in substitution point sets, which is a concept parallel to overlap coincidence, and in Section 3 show the equivalence between (1) and (2) of the theorem. Proposition 3.10 plays an important role in making connection between algebraic coincidence and pure point spectrum. In Section 4, we assume algebraic coincidence in $\mathbf{\Lambda}$ and construct a CPS whose internal space is a completion of a topological group L with the Q-topology. We then show that there exists a Delone multi-colour point set $\mathbf{\Gamma}$, in a local dynamical hull generated by $\mathbf{\Lambda}$ which is a model set with an open window in the CPS. We consider another topology (P_ϵ-topology) on L relative to which L becomes a topological group and show in Section 4.4 that the two topological spaces L are in fact isomorphic. So both topologies lead to the same completion of L. It is a locally compact Abelian group and we can construct a CPS taking this completed space as an internal space. In Section 4.5 we apply the results of [3,21], which are associated with the P_ϵ-topology, so that we get a condition for $\mathbf{\Lambda}$ to be an inter-model multi-colour set. We observe that algebraic coincidence is sufficient for that condition to be fulfilled in substitution point sets. We also show that the algebraic coincidence is necessary to obtain the inter-model multi-colour set in Section 5.

The paper concludes with some unresolved questions (particularly on the nature of the boundaries of the windows of the inter-model sets appearing in the theorem) and outlook for future work.

2. Preliminaries

Much of the terminology being introduced in this section is standard and defined precisely in [23]. We refer the reader to [23] for more detailed definitions and to [17] for the standard concepts of discrete geometry in the aperiodic setting.

2.1. Delone multi-colour sets

A *multi-colour set* or *m-multi-colour set* in $\mathbb{R}^d$ is a subset $\mathbf{\Lambda} = \Lambda_1 \times \cdots \times \Lambda_m \subset \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ (m copies) where $\Lambda_i \subset \mathbb{R}^d$. We also write $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_m) = (\Lambda_i)_{i \le m}$. Recall that a Delone set is a relatively dense and uniformly discrete subset of $\mathbb{R}^d$. We say that $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is a *Delone multi-colour set* in $\mathbb{R}^d$ if each Λ_i is Delone and $\mathrm{supp}(\mathbf{\Lambda}) := \bigcup_{i=1}^m \Lambda_i \subset \mathbb{R}^d$ is Delone. A *cluster* of $\mathbf{\Lambda}$ is, by definition, a family $\mathbf{P} = (P_i)_{i \le m}$ where $P_i \subset \Lambda_i$ is finite for all $i \le m$. Many of the clusters that we consider have the form $\mathbf{\Lambda} \cap A := (\Lambda_i \cap A)_{i \le m}$, for a bounded set $A \subset \mathbb{R}^d$. The translate of a cluster $\mathbf{P}$ by $x \in \mathbb{R}^d$ is $x + \mathbf{P} = (x + P_i)_{i \le m}$. We say that two clusters $\mathbf{P}$ and $\mathbf{P}'$ are *translationally equivalent* if $\mathbf{P} = x + \mathbf{P}'$ for some $x \in \mathbb{R}^d$. For any two Delone m-multi-colour sets $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$, we define $\mathbf{\Lambda} \cap \mathbf{\Gamma} = (\Lambda_i \cap \Gamma_i)_{i \le m}$ and $\mathbf{\Lambda} \triangle \mathbf{\Gamma} = (\Lambda_i \triangle \Gamma_i)_{i \le m}$, where $\Lambda_i \triangle \Gamma_i = (\Lambda_i \setminus \Gamma_i) \cup (\Gamma_i \setminus \Lambda_i)$. We write $B_R(y)$ for the open ball of radius R centered at y and use also B_R for $B_R(0)$. We define $\Xi(\mathbf{\Lambda}) := \bigcup_{i \le m} (\Lambda_i - \Lambda_i)$. We say that $\Lambda \subset \mathbb{R}^d$ is a *Meyer set* if it is a Delone set and $\Lambda - \Lambda \subset \Lambda + F$ for some finite set F, equivalently, if it is a Delone set and $\Lambda - \Lambda$ is uniformly discrete (see [16,25]). We say $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ a *Meyer multi-colour set* if each component Λ_i, $i \le m$, is a Meyer set and $\mathrm{supp}(\mathbf{\Lambda})$ is a Meyer set. A multi-colour set $\mathbf{\Lambda}$ is said to be *non-periodic* if there is no non-zero $x \in \mathbb{R}^d$ such that $\mathbf{\Lambda} + x = \mathbf{\Lambda}$.

A *cut and project scheme* (CPS) consists of a collection of spaces and mappings as follows;

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \bigcup & & \\ & & \widetilde{L} & & \end{array} \tag{2.1}$$

where $\mathbb{R}^d$ is a real Euclidean space, H is some locally compact Abelian group, π_1 and π_2 are the canonical projections, $\widetilde{L} \subset \mathbb{R}^d \times H$ is a lattice, i.e. a discrete subgroup for which the quotient group $(\mathbb{R}^d \times H)/\widetilde{L}$ is compact, $\pi_1|_{\widetilde{L}}$ is injective, and $\pi_2(\widetilde{L})$ is dense in H.

We call $\mathbb{R}^d$ a *physical space* and H an *internal space*. For a subset $V \subset H$, we denote $\Lambda(V) := \{\pi_1(x) \in \mathbb{R}^d : x \in \widetilde{L}, \pi_2(x) \in V\}$. We call the subset V a *window* of $\Lambda(V)$. A *model set* in $\mathbb{R}^d$ is a subset Γ of $\mathbb{R}^d$ for which $\Gamma = \Lambda(V)$ where $V \subset H$ has non-empty interior and compact closure. The model set Γ is *regular* if the boundary $\partial W = \overline{W} \setminus W^\circ$ of W is of (Haar) measure 0.

Definition 2.1. An *inter-model set* is a subset Γ of $\mathbb{R}^d$ for which $s + \Lambda(W^\circ) \subset \Gamma \subset s + \Lambda(W)$ for some $s \in \mathbb{R}^d$, where W is compact in H and $W = \overline{W^\circ} \neq \emptyset$, with respect to CPS (2.1).

We say that $\mathbf{\Gamma}$ is a *model multi-colour set* (resp. *inter-model multi-colour set*) if each Γ_i is a model set (resp. inter-model set) with respect to the same CPS (see [26,21,3] for more about model sets). One should note here that since π_2 need not be $1-1$ on $\widetilde{L}$ in CPS, the notion of inter-model set, which is hemmed in between two such sets differing only by points on the boundary of the window W, arises naturally. When it is important to note which CPS a model set arises from, we will explicitly mention the CPS.

Let $\mathbf{\Lambda}$ be a Delone multi-colour set. We say that $\mathbf{\Lambda}$ has *finite local complexity (FLC)* if for every $R > 0$ there exists a finite set $Y \subset \mathrm{supp}(\mathbf{\Lambda}) = \bigcup_{i=1}^m \Lambda_i$ such that for all $x \in \mathrm{supp}(\mathbf{\Lambda})$, there exists $y \in Y$ for which $B_R(x) \cap \mathbf{\Lambda} = (B_R(y) \cap \mathbf{\Lambda}) + (x - y)$. Also we say that $\mathbf{\Lambda}$ is *repetitive* if for every compact set $K \subset \mathbb{R}^d$, $\{t \in \mathbb{R}^d : \mathbf{\Lambda} \cap K = (t + \mathbf{\Lambda}) \cap K\}$ is relatively dense. For a cluster $\mathbf{P}$ and a bounded set $A \subset \mathbb{R}^d$, let

$$L_{\mathbf{P}}(A) := \sharp\{x \in \mathbb{R}^d : x + \mathbf{P} \subset A \cap \mathbf{\Lambda}\},$$

where $\sharp$ means the cardinality. A *van Hove sequence* for $\mathbb{R}^d$ is a sequence $\mathcal{F} = \{F_n\}_{n \geq 1}$ of bounded measurable subsets of $\mathbb{R}^d$ satisfying

$$\lim_{n \to \infty} \mathrm{Vol}((\partial F_n)^{+r})/\mathrm{Vol}(F_n) = 0, \quad \text{for all } r > 0, \tag{2.2}$$

where $(\partial F_n)^{+r} := \{x \in \mathbb{R}^d : \mathrm{dist}(x, \partial F_n) \leq r\}$. We define

$$\mathrm{dens}(\mathbf{\Lambda}) := \lim_{n \to \infty} \frac{\sharp(\mathbf{\Lambda} \cap F_n)}{\mathrm{Vol}(F_n)},$$

if the limit exists. We say that $\mathbf{\Lambda}$ has *uniform cluster frequencies*[1] (UCF) relative to $\{F_n\}_{n \geq 1}$ if for any cluster $\mathbf{P}$, there exists the limit

$$\mathrm{freq}(\mathbf{P}, \mathbf{\Lambda}) = \lim_{n \to \infty} \frac{L_{\mathbf{P}}(x + F_n)}{\mathrm{Vol}(F_n)},$$

uniformly in $x \in \mathbb{R}^d$.

Let $X_{\mathbf{\Lambda}}$ be the collection of all Delone multi-colour sets each of whose clusters is a translate of a $\mathbf{\Lambda}$-cluster. We introduce a metric on Delone multi-colour sets in a simple variation of the standard way: for Delone multi-colour sets $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in X_{\mathbf{\Lambda}}$,

$$d(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2) := \min\{\tilde{d}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2), 2^{-1/2}\}, \tag{2.3}$$

where

$$\tilde{d}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2) = \inf\{\varepsilon > 0 : \exists x, y \in B_\varepsilon(0), B_{1/\varepsilon}(0) \cap (-x + \mathbf{\Lambda}_1) = B_{1/\varepsilon}(0) \cap (-y + \mathbf{\Lambda}_2)\}.$$

For the proof that d is a metric, see [22]. Observe that $X_{\mathbf{\Lambda}} = \overline{\{-h + \mathbf{\Lambda} : h \in \mathbb{R}^d\}}$ where the closure is taken in the topology induced by the metric d. For more general topology defined by uniformity, see [21,29,3]. We have a natural action of $\mathbb{R}^d$ on the dynamical hull $X_{\mathbf{\Lambda}}$ of $\mathbf{\Lambda}$ by translations which makes it a topological dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$. With FLC, $X_{\mathbf{\Lambda}}$ is a compact space.

[1] Here we define UCF with one fixed van Hove sequence. However it is implicit from [22] that UCF for the Delone multi-colour set with FLC does not depend on the choice of van Hove sequence.

Let $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ be a measure preserving dynamical system. We consider the associated group of unitary operators $\{T_x\}_{x\in\mathbb{R}^d}$ on $L^2(X_{\mathbf{\Lambda}}, \mu)$:

$$T_x g(\mathbf{\Lambda}') = g(-x + \mathbf{\Lambda}').$$

Every $g \in L^2(X_{\mathbf{\Lambda}}, \mu)$ defines a function on $\mathbb{R}^d$ by $x \mapsto \langle T_x g, g\rangle$. This function is positive definite on $\mathbb{R}^d$, so its Fourier transform is a positive measure σ_g on $\mathbb{R}^d$ called the *spectral measure* corresponding to g. The dynamical system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ is said to have *pure point spectrum* if σ_g is pure point for every $g \in L^2(X_{\mathbf{\Lambda}}, \mu)$.[2] We recall that $g \in L^2(X_{\mathbf{\Lambda}}, \mu)$ is an eigenfunction for the $\mathbb{R}^d$-action if for some $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$,

$$T_x g = \mathrm{e}^{2\pi \mathrm{i} x\cdot\alpha} g \quad \text{for all } x \in \mathbb{R}^d,$$

where $x \cdot \alpha$ is the standard inner product on $\mathbb{R}^d$.

2.2. Tilings

We begin with a set of types (or colours) $\{1, \ldots, m\}$, which we fix once and for all. A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$ (the support of T) is a compact set in $\mathbb{R}^d$, which is the closure of its interior, and $i = l(T) \in \{1, \ldots, m\}$ is the type of T. We let $g + T = (g + A, i)$ for $g \in \mathbb{R}^d$. We say that a set P of tiles is a *patch* if the number of tiles in P is finite and the tiles of P have mutually disjoint interiors. The *support of a patch* is the union of the supports of the tiles that are in it. The *translate of a patch* P by $g \in \mathbb{R}^d$ is $g + P := \{g + T : T \in P\}$. We say that two patches P_1 and P_2 are *translationally equivalent* if $P_2 = g + P_1$ for some $g \in \mathbb{R}^d$. A tiling of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \bigcup\{\mathrm{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles have disjoint interiors. Given a tiling $\mathcal{T}$, a finite set of tiles of $\mathcal{T}$ is called $\mathcal{T}$-patch. We define FLC, repetitivity, and uniform patch frequencies (UPF), which is the analog of UCF, on tilings in the same way as the corresponding properties on Delone multi-colour sets. The types (or colours) of tiles on tilings have the same concept as the colours of points on Delone multi-colour sets. We always assume that any two $\mathcal{T}$-tiles with the same colour are translationally equivalent (hence there are finitely many $\mathcal{T}$-tiles up to translations).

For a subset $\mathcal{S}$ of a tiling and $A \subset \mathbb{R}^d$, we define

$$\mathcal{S} \cap A := \{T \in \mathcal{S} : (\mathrm{supp}(T))^\circ \cap A \neq \emptyset\}$$

and for tilings $\mathcal{T}$ and $\mathcal{T}'$, we use $\mathcal{T} \cap \mathcal{T}' \cap A$ for $(\mathcal{T} \cap \mathcal{T}') \cap A$. For any patch P, we write $\mathrm{Vol}(P)$ for $\mathrm{Vol}(\bigcup\{\mathrm{supp}(T) : T \in P\})$. Just as for point sets, for any $D \subset \mathcal{T}$, we define

$$\mathrm{dens}(D) := \lim_{n\to\infty} \frac{\mathrm{Vol}(D \cap F_n)}{\mathrm{Vol}(F_n)} \quad \text{and} \quad \mathrm{freq}(P, \mathcal{T}) := \lim_{n\to\infty} \frac{L_P(F_n)}{\mathrm{Vol}(F_n)},$$

if the limits exist. Let $X_{\mathcal{T}}$ be the collection of all tilings each of whose patches is a translate of $\mathcal{T}$-patch. We define a metric d on tilings, given analogously to (2.3) for Delone multi-colour sets: for tilings $\mathcal{T}, \mathcal{S} \in X_{\mathcal{T}}$,

$$d(\mathcal{T}, \mathcal{S}) := \min\{\tilde{d}(\mathcal{T}, \mathcal{S}), 2^{-1/2}\}, \tag{2.4}$$

where

$$\tilde{d}(\mathcal{T}, \mathcal{S}) = \inf\{\varepsilon > 0 : \exists x, y \in B_\varepsilon(0), (-x + \mathcal{T}) \cap B_{1/\varepsilon}(0) = (-y + \mathcal{S}) \cap B_{1/\varepsilon}(0)\}.$$

We define the dynamical hull $(X_{\mathcal{T}}, \mathbb{R}^d)$ of $\mathcal{T}$ in the same way as of Delone multi-colour sets (see [31]). Also we have the equivalent notion of pure point spectrum on tilings.

2.3. Substitutions

2.3.1. Substitutions on Delone multi-colour sets

We say that a linear map $Q : \mathbb{R}^d \to \mathbb{R}^d$ is *expansive* if there is a $c > 1$ with

$$e(Qx, Qy) \geq c \cdot e(x, y) \tag{2.5}$$

[2] We also say that $\mathbf{\Lambda}$ has pure point spectrum if σ_g is pure point for every $g \in L^2(X_{\mathbf{\Lambda}}, \mu)$.

for all $x, y \in \mathbb{R}^d$ and some metric e on $\mathbb{R}^d$ compatible with the standard topology. This is equivalent to saying that all the eigenvalues of Q lie outside the closed unit disk in $\mathbb{C}$.

Definition 2.2. $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is called a *substitution Delone multi-colour set* if $\mathbf{\Lambda}$ is a Delone multi-colour set and there exist an expansive map $Q : \mathbb{R}^d \to \mathbb{R}^d$ and finite sets $\mathcal{D}_{ij}$ for $i, j \le m$ such that

$$\Lambda_i = \bigcup_{j=1}^{m} (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m, \tag{2.6}$$

where the unions on the right-hand side are disjoint.

We say that the substitution Delone multi-colour set is *primitive* if the corresponding substitution matrix S, with $S_{ij} = \sharp(\mathcal{D}_{ij})$, is primitive. For any given substitution Delone multi-colour set $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$, we define $\Phi_{ij} = \{f : x \mapsto Qx + a : a \in \mathcal{D}_{ij}\}$. Then $\Phi_{ij}(\Lambda_j) = Q\Lambda_j + \mathcal{D}_{ij}$, where $i \le m$. We define Φ an $m \times m$ array for which each entry is Φ_{ij}, and call Φ a *matrix function system (MFS)* for the substitution. For any $k \in \mathbb{Z}_+$ and $x \in \Lambda_j$ with $j \le m$, we let $\Phi^k(x) = \Phi^{k-1}((\Phi_{ij}(x))_{i \le m})$. For any $k \in \mathbb{Z}_+$, $\Phi^k(\mathbf{\Lambda}) = \mathbf{\Lambda}$ and $\Phi^k(\Lambda_j) = \bigcup_{i \le m}(Q^k\Lambda_j + (\mathcal{D}^k)_{ij})$ where

$$(\mathcal{D}^k)_{ij} = \bigcup_{n_1, n_2, \ldots, n_{(k-1)} \le m} (\mathcal{D}_{in_1} + Q\mathcal{D}_{n_1 n_2} + \cdots + Q^{k-1}\mathcal{D}_{n_{(k-1)} j}).$$

We say that a cluster $\mathbf{P}$ is *legal* if it is a translate of a subcluster of a cluster generated from one point of $\mathbf{\Lambda}$, i.e. $a + \mathbf{P} \subset \Phi^k(x)$ for some $k \in \mathbb{Z}_+$, $a \in \mathbb{R}^d$ and $x \in \mathbf{\Lambda}$.

2.3.2. *Substitutions on tilings*

Definition 2.3. Let $\mathcal{A} = \{T_1, \ldots, T_m\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles*. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches made of tiles each of which is a translate of one of T_i's. We say that $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is a *tile-substitution* (or simply *substitution*) with expansive map Q if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le m$, such that

$$\omega(T_j) = \{u + T_i : u \in \mathcal{D}_{ij},\ i = 1, \ldots, m\} \tag{2.7}$$

with

$$QA_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i) \quad \text{for } j \le m.$$

Here all sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty.

Note that $QA_j = \mathrm{supp}(\omega(T_j)) = Q\mathrm{supp}(T_j)$. The substitution (2.7) is extended to all translates of prototiles by

$$\omega(x + T_j) = Qx + \omega(T_j), \tag{2.8}$$

in particular,

$$\mathrm{supp}(\omega(x + T_j)) = \mathrm{supp}(Qx + \omega(T_j)) = Qx + Q\mathrm{supp}(T_j) = Q(x + \mathrm{supp}(T_j)), \tag{2.9}$$

and to patches and tilings by $\omega(P) = \cup\{\omega(T) : T \in P\}$. The substitution ω can be iterated, producing larger and larger patches $\omega^k(T_j)$.

We define the substitution matrix and primitivity of ω in the similar way as in substitution Delone multi-colour sets. We say that $\mathcal{T}$ is a *substitution tiling* if $\mathcal{T}$ is a tiling and $\omega(\mathcal{T}) = \mathcal{T}$ with some substitution ω. We say that a patch P is *legal* if it is a translate of a subpatch of $\omega^k(T_i)$ for some $i \le m$ and $k \ge 1$. This is the analog of a legal cluster on Delone multi-colour sets.

2.3.3. Representability of $\mathbf{\Lambda}$ *as a tiling*

Let $\mathbf{\Lambda}$ be a substitution Delone multi-colour set. One can set up an *adjoint system* of equations

$$QA_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i), \quad j \le m \tag{2.10}$$

from the Eq. (2.6). It is known that (2.10) always has a unique solution for which $\mathcal{A} = \{A_1, \ldots, A_m\}$ is a family of non-empty compact sets of $\mathbb{R}^d$ (see for example [5], Prop. 1.3). It is proved in [18, Th. 2.4 and Th. 5.5] that under the condition of primitivity, all the sets A_i from (2.10) have non-empty interiors and, moreover, each A_i is the closure of its interior. We say that $\mathbf{\Lambda}$ is *representable* (by tiles) if $\mathbf{\Lambda} + \mathcal{A} := \{x + T_i : x \in \Lambda_i, i \le m\}$ is a tiling of $\mathbb{R}^d$, where $T_i = (A_i, i)$, $i \le m$, and A_i's arise from the solution to the adjoint system (2.10) and $\mathcal{A} = \{T_i : i \le m\}$. One can define a tile-substitution ω satisfying $\omega(\mathbf{\Lambda} + \mathcal{A}) = \mathbf{\Lambda} + \mathcal{A}$ from (2.10). So $\mathbf{\Lambda} + \mathcal{A}$ is a substitution tiling. We call $\mathbf{\Lambda} + \mathcal{A}$ the *associated substitution tiling* of $\mathbf{\Lambda}$. In [18, Lemma 3.2] it is shown that if $\mathbf{\Lambda}$ is a substitution Delone multi-colour set, then there is a finite multi-colour set (cluster) $\mathbf{P} \subset \mathbf{\Lambda}$ for which $\Phi^{n-1}(\mathbf{P}) \subset \Phi^n(\mathbf{P})$ for $n \ge 1$ and $\mathbf{\Lambda} = \lim_{n\to\infty} \Phi^n(\mathbf{P})$. We call such a multi-colour set $\mathbf{P}$ a *generating set* for $\mathbf{\Lambda}$.

Theorem 2.4 (*[23]*). *Let* $\mathbf{\Lambda}$ *be a repetitive primitive substitution Delone multi-colour set. Then every* $\mathbf{\Lambda}$*-cluster is legal if and only if* $\mathbf{\Lambda}$ *is representable.*

Note that in order to check that every $\mathbf{\Lambda}$-cluster is legal, it suffices to check if a cluster containing a finite generating set for $\mathbf{\Lambda}$ is legal (see [23]).

Remark 2.5. Throughout this paper we are mainly interested in primitive substitution Delone multi-colour sets $\mathbf{\Lambda}$ such that every $\mathbf{\Lambda}$-cluster is legal. Since $\mathbf{\Lambda}$ is representable, we will often identify $\mathbf{\Lambda}$ with the associated substitution tilings $\mathcal{T} = \mathbf{\Lambda} + \mathcal{A} = \{x + T_i : x \in \Lambda_i, i \le m\}$, where T_i's are the tiles arising from the solution of the adjoint system of equations.

For a primitive representable substitution Delone multi-colour set $\mathbf{\Lambda}$, the dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ is unique ergodic. Similarly the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ of any primitive substitution tiling $\mathcal{T}$ is uniquely ergodic [23].

3. Algebraic coincidence and pure point spectrum

For substitution tilings overlap coincidence was introduced in [31]. The overlap coincidence has been a central concept connecting the asymptotic behavior of Q-iterates of almost-periods of tilings with pure pointedness of the spectrum of dynamical systems generated by the tilings (see [31,23]). Here we introduce a new coincidence condition for substitution Delone multi-colour sets and show that it is equivalent to the overlap coincidence when the substitution Delone multi-colour sets are representable.

Let $\mathcal{T}$ be a tiling and $\Xi(\mathcal{T})$ be the set of translation vectors between $\mathcal{T}$-tiles of the same type:

$$\Xi(\mathcal{T}) := \{x \in \mathbb{R}^d : \exists T, T' \in \mathcal{T}, T' = x + T\}. \tag{3.1}$$

Since $\mathcal{T}$ has the inflation symmetry with the expansive map Q, we have that $Q\Xi(\mathcal{T}) \subset \Xi(\mathcal{T})$. Note also that $\Xi(\mathcal{T}) = -\Xi(\mathcal{T})$. If $\mathcal{T} = \mathbf{\Lambda} + \mathcal{A}$ is an associated substitution tiling of $\mathbf{\Lambda}$, then $\Xi(\mathcal{T}) = \bigcup_{i=1}^{m} (\Lambda_i - \Lambda_i)$. Recall that $\omega\mathcal{T} = \mathcal{T}$, where ω is the tile-substitution coming from the adjoint system of equations.

Definition 3.1. Let $\mathcal{T}$ be a tiling. A triple (T, y, S), with $T, S \in \mathcal{T}$ and $y \in \Xi(\mathcal{T})$, is called an *overlap* if $\mathrm{supp}(y + T) \cap \mathrm{supp}(S)$ has non-empty interior. We say that two overlaps (T, y, S) and (T', y', S') are *equivalent* if for some $g \in \mathbb{R}^d$ we have $y + T = g + y' + T'$, $S = g + S'$. Denote by $[(T, y, S)]$ the equivalence class of an overlap. An overlap (T, y, S) is a *coincidence* if $y + T = S$. The *support* of an overlap (T, y, S) is $\mathrm{supp}(T, y, S) = \mathrm{supp}(y + T) \cap \mathrm{supp}(S)$.

Let $\mathcal{O} = (T, y, S)$ be an overlap. Recall that for a tile-substitution ω, $\omega(y+T) = Qy+\omega(T)$ is a patch of $Qy+\mathcal{T}$, and $\omega(S)$ is a $\mathcal{T}$-patch, and moreover,

$$\mathrm{supp}(Qy + \omega(T)) \cap \mathrm{supp}(\omega(S)) = Q(\mathrm{supp}(T, y, S)).$$

For each $l \in \mathbb{Z}_+$,

$$Q^l(\mathcal{O}) = \{(T', Q^l y, S') : T' \in \omega^l(T), S' \in \omega^l(S), \operatorname{supp}(Q^l y + T') \cap \operatorname{supp}(S') \neq \emptyset\}.$$

Definition 3.2. We say that a substitution tiling $\mathcal{T}$ admits an *overlap coincidence* if there exists $l \in \mathbb{Z}_+$ such that for each overlap $\mathcal{O}$ in $\mathcal{T}$, $Q^l(\mathcal{O})$ contains a coincidence.

Theorem 3.3 (*[23, Th. 4.7 and Lemma A.9]*). *Let $\mathcal{T}$ be a repetitive fixed point of a primitive substitution such that $\Xi(\mathcal{T})$ is a Meyer set. Then $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ has a pure point dynamical spectrum if and only if $\mathcal{T}$ admits an overlap coincidence.*

Theorem 3.4 (*[24]*). *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal and $\mathbf{\Lambda}$ has FLC. Suppose that $(X_{\mathbf{\Lambda}}, \mathbb{R}^d, \mu)$ has a pure point dynamical spectrum. Then $\Lambda = \bigcup_{i \le m} \Lambda_i$ and $\Xi(\mathbf{\Lambda})$ are Meyer sets.*

From the result of Theorem 3.4, we can get the following corollary of Theorem 3.3 dropping the Meyer condition.

Corollary 3.5. *Let $\mathcal{T}$ be a repetitive fixed point of a primitive substitution with FLC. Then $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ has a pure point dynamical spectrum if and only if $\mathcal{T}$ admits an overlap coincidence.*

Proof. We first note that when we replace the third condition of [23, Lemma A.9] by the overlap coincidence that we define here, Lemma A.9 in [23] holds without the assumption of the Meyer property. So applying [31, Th. 6.1], we can see that the necessity direction holds. The sufficiency follows from Theorems 3.3 and 3.4. □

Combining Theorem 3.4 and Corollary 3.5, we observe the following.

Corollary 3.6. *Let $\mathcal{T}$ be a repetitive fixed point of a primitive substitution with FLC. If $\mathcal{T}$ admits an overlap coincidence, then $\Xi(\mathcal{T})$ is a Meyer set.* □

Lemma 3.7 (*[23, Lemma A.8]*). *Let $\mathcal{T}$ be a tiling such that $\Xi(\mathcal{T})$ is a Meyer set. Then the number of equivalence classes of overlaps for $\mathcal{T}$ is finite.*

Definition 3.8. Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set with expansive map Q. We say that $\mathbf{\Lambda}$ admits an *algebraic coincidence* if there exist $M \in \mathbb{Z}_+$ and $\xi \in \Lambda_i$ for some $i \le m$ such that $\xi + Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_i$. We say that $\mathbf{\Lambda}$ admits an *algebraic coincidence at* ξ, when we need to emphasize the role of ξ for the algebraic coincidence.

Lemma 3.9. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set with expansive map Q. If $\mathbf{\Lambda}$ admits an algebraic coincidence, then $\Xi(\mathbf{\Lambda})$ is a Meyer set and thus $\mathbf{\Lambda}$ has FLC.*

Proof. By the assumption, there exist $M \in \mathbb{Z}_+$ and $\xi \in \Lambda_i$ for some $i \le m$ such that $\xi + Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_i$. Since $\mathbf{\Lambda}$ is a Delone multi-colour set, $Q^M \Xi(\mathbf{\Lambda})$ is uniformly discrete. So $\Xi(\mathbf{\Lambda})$ is uniformly discrete. Note that $Q^M \Xi(\mathbf{\Lambda}) - Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_i - \Lambda_i \subset \Xi(\mathbf{\Lambda})$. Thus $\Xi(\mathbf{\Lambda}) - \Xi(\mathbf{\Lambda})$ is uniformly discrete, i.e. $\Xi(\mathbf{\Lambda})$ is a Meyer set. Then it is easy to see that $\mathbf{\Lambda}$ has FLC. □

We now connect overlap coincidence with algebraic coincidence. For this connection, the condition of $\Xi(\mathbf{\Lambda})$ being a Meyer set is strongly used.

Proposition 3.10. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal and $\mathbf{\Lambda}$ has FLC. Suppose that the associated substitution tiling $\mathcal{T} = \mathbf{\Lambda} + \mathcal{A}$ admits an overlap coincidence. Then $\mathbf{\Lambda}$ admits an algebraic coincidence.*

Proof. We first sketch the idea of the proof. First, we note that $\Xi(\mathcal{T})$ is a Meyer set by Corollary 3.6. From Lemma 3.7, there are only finitely many possible overlaps in $\mathcal{T}$. We start with any non-empty patch P in $\mathcal{T}$. As $\mathcal{T}$ is translated by the vectors in $\Xi(\mathcal{T})$ and intersects with $\mathcal{T}$, we get certain configurations of overlaps on $\operatorname{supp}(P)$. There are only finitely many possible configurations of overlaps on $\operatorname{supp}(P)$ that arise this way. When each configuration of overlaps on $\operatorname{supp}(P)$ is enlarged enough by applying the substitution repeatedly, there are many coincidences of overlaps occurring in the enlarged configuration. Ultimately they cover most of the volume of the enlarged configuration, due to the assumption of overlap coincidence (see [23, Lemma A.9]). But the number of configurations of overlaps on $\mathrm{supp}(P)$ stays the same as we enlarge them. So when we intersect all the coincidences of overlaps in the enlarged configuration for all the translational vectors in $\Xi(\mathcal{T})$, we get a *non-empty* set. This implies that there exists at least one tile in $\mathcal{T}$ whose translations by the translational vectors of $Q^M \Xi(\mathcal{T})$ are all in $\mathcal{T}$. This implies algebraic coincidence for $\mathbf{\Lambda}$.

Now we give a detailed proof. We consider the associated substitution tiling $\mathcal{T} = \mathbf{\Lambda} + \mathcal{A}$ of $\mathbf{\Lambda}$ and choose any non-empty patch P in $\mathcal{T}$. Note that $\Xi(\mathcal{T}) = \Xi(\mathbf{\Lambda})$. Consider the collection of patches of translates of $\mathcal{T}$ on $\mathrm{supp}(P)$.

$$\mathcal{H} = \{(\alpha + \mathcal{T}) \cap \mathrm{supp}(P) : \alpha \in \Xi(\mathcal{T})\}.$$

Since $\Xi(\mathcal{T})$ is a Meyer set, the number of translationally equivalent classes of overlaps for $\mathcal{T}$ is finite by Lemma 3.7. It is important to note as a result of this that $\mathcal{H}$ consists of only finitely many patches. Notice that this is more than just saying that there are finitely many translational classes of patches, which simply means the FLC of $\mathbf{\Lambda}$. Thus we can find $\alpha_1, \dots, \alpha_K \in \Xi(\mathcal{T})$ such that for any $\alpha \in \Xi(\mathcal{T})$,

$$(\alpha + \mathcal{T}) \cap \mathrm{supp}(P) = (\alpha_k + \mathcal{T}) \cap \mathrm{supp}(P) \quad \text{for some } k \le K. \tag{3.2}$$

For any $n \in \mathbb{Z}_+$, we get

$$\omega^n((\alpha + \mathcal{T}) \cap \mathrm{supp}(P)) = \omega^n((\alpha_k + \mathcal{T}) \cap \mathrm{supp}(P)). \tag{3.3}$$

Looking at the set of (3.3) on the compact set $Q^n \mathrm{supp}(P)$, we get from $\omega^n \mathcal{T} = \mathcal{T}$ and (2.8) that

$$(Q^n \alpha + \mathcal{T}) \cap Q^n \mathrm{supp}(P) = (Q^n \alpha_k + \mathcal{T}) \cap Q^n \mathrm{supp}(P).$$

Then

$$\mathcal{T} \cap (Q^n \alpha + \mathcal{T}) \cap Q^n \mathrm{supp}(P) = \mathcal{T} \cap (Q^n \alpha_k + \mathcal{T}) \cap Q^n \mathrm{supp}(P). \tag{3.4}$$

Let

$$D_{Q^n \alpha} := \mathcal{T} \cap (Q^n \alpha + \mathcal{T}) \cap Q^n \mathrm{supp}(P), \tag{3.5}$$

for any $\alpha \in \Xi(\mathcal{T})$. Then from (3.2),

$$\bigcap_{\alpha \in \Xi(\mathcal{T})} D_{Q^M \alpha} = \bigcap_{k=1}^{K} D_{Q^M \alpha_k}. \tag{3.6}$$

We claim that there exists $M \in \mathbb{Z}_+$ such that

$$\bigcap_{k=1}^{K} D_{Q^M \alpha_k} \neq \emptyset. \tag{3.7}$$

Then it implies that there exists $T \in \bigcap_{\alpha \in \Xi(\mathcal{T})} D_{Q^M \alpha}$ from (3.6) and so for any $\alpha \in \Xi(\mathcal{T})$ there exists $T' \in \mathcal{T}$ such that $Q^M \alpha + T' = T$. Note that $T = \xi + T_i$ and $T' = \eta + T_i$ where $\xi, \eta \in \Lambda_i$ for some $i \le m$. So $\xi - Q^M \alpha \in \Lambda_i$ and we can conclude that $\xi - Q^M \Xi(\mathbf{\Lambda}) = \xi + Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_i$. Therefore $\mathbf{\Lambda}$ admits an algebraic coincidence.

Now we give the proof of the claim (3.7). We use a type of argument, leading to (3.10), that has been used before in [31,23]. However it is not in the form that we can make direct use here, so we discuss it again in the form that we need. Let V_0 and V_1 be the minimal and maximal volumes of $\mathcal{T}$-tiles respectively. Notice that $\mathrm{supp}(D_{Q^n \alpha})$ is the union of supports of coincidences in $Q^n \mathrm{supp}(P)$. It is easy to see that coincidence in $D_{Q^n \alpha}$ leads to other coincidences in $D_{Q^{n+1} \alpha}$. Thus

$$Q \mathrm{supp}(D_{Q^n \alpha}) \subset \mathrm{supp}(D_{Q^{n+1} \alpha}).$$

Since $\mathcal{T}$ admits an overlap coincidence, there exists $l \in \mathbb{Z}_+$ such that for each overlap $\mathcal{O}$ in $\mathcal{T}$, $Q^l(\mathcal{O})$ contains a coincidence. Note that $\mathrm{supp}(D_{Q^{n+l} \alpha})$ has more support than $Q^l \mathrm{supp}(D_{Q^n \alpha})$ from the new coincidence occurring after

l-step iterations of each non-coincidence overlap, and the volume of the support of the new coincidence is at least V_0 for every l-step iteration of each non-coincidence overlap. So we can get the following formula

$$\mathrm{Vol}(D_{Q^{n+l}\alpha}) - |\det Q|^l \mathrm{Vol}(D_{Q^n\alpha}) \geq \frac{V_0}{V_1 |\det Q|^l} \left(\mathrm{Vol}(Q^{n+l}\mathrm{supp}(P)) - |\det Q|^l \mathrm{Vol}(D_{Q^n\alpha})\right). \tag{3.8}$$

Letting

$$b = 1 - \frac{V_0}{V_1 |\det Q|^l}$$

and using

$$\mathrm{Vol}(Q^{n+l}\mathrm{supp}(P)) = |\det Q|^l \mathrm{Vol}(Q^n \mathrm{supp}(P)),$$

the inequality (3.8) becomes

$$1 - \frac{\mathrm{Vol}(D_{Q^{n+l}\alpha})}{\mathrm{Vol}(Q^{n+l}\mathrm{supp}(P))} \leq b\left(1 - \frac{\mathrm{Vol}(D_{Q^n\alpha})}{\mathrm{Vol}(Q^n\mathrm{supp}(P))}\right). \tag{3.9}$$

For all $n = tl + s \geq 0$ where $t \in \mathbb{Z}_+$ and $0 \leq s < l$, we obtain from (3.9)

$$\begin{aligned} 1 - \frac{\mathrm{Vol}(D_{Q^n\alpha})}{\mathrm{Vol}(Q^n\mathrm{supp}(P))} &\leq b^t\left(1 - \frac{\mathrm{Vol}(D_{Q^s\alpha})}{\mathrm{Vol}(Q^s\mathrm{supp}(P))}\right) \\ &= (b^{1/l})^{tl+s}\frac{1}{b^{s/l}}\left(1 - \frac{\mathrm{Vol}(D_{Q^s\alpha})}{\mathrm{Vol}(Q^s\mathrm{supp}(P))}\right) \\ &\leq r^n c \quad \text{for some } r \in (0,1) \text{ and } c > 0. \end{aligned} \tag{3.10}$$

Thus for any $\epsilon > 0$, we can find $M \in \mathbb{Z}_+$ such that for any $1 \leq k \leq K$,

$$1 - \frac{\mathrm{Vol}(D_{Q^M\alpha_k})}{\mathrm{Vol}(Q^M\mathrm{supp}(P))} < \epsilon.$$

This implies that

$$1 - \frac{\mathrm{Vol}\left(\bigcap_{k=1}^{K} D_{Q^M\alpha_k}\right)}{\mathrm{Vol}(Q^M\mathrm{supp}(P))} < \epsilon K.$$

Therefore for small $\epsilon > 0$

$$\bigcap_{k=1}^{K} D_{Q^M\alpha_k} \neq \emptyset,$$

as we claimed in (3.7). □

We show now the converse direction of Proposition 3.10. From Lemma 3.9, we do not need to additionally assume the Meyer property of $\Xi(\mathbf{\Lambda})$ in the following proposition.

Proposition 3.11. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal. Suppose that $\mathbf{\Lambda}$ admits an algebraic coincidence. Then the associated substitution tiling $\mathcal{T} = \mathbf{\Lambda} + \mathcal{A}$ admits an overlap coincidence.*

Proof. Suppose that there exist $M \in \mathbb{Z}_+$ and $\xi \in \Lambda_i$ such that $\xi + Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_i$ for some $i \leq m$. Then

$$\begin{aligned} Q^M\Xi(\mathbf{\Lambda}) + Q^M\Xi(\mathbf{\Lambda}) &= Q^M\Xi(\mathbf{\Lambda}) - Q^M\Xi(\mathbf{\Lambda}) \\ &\subset (\Lambda_i - \xi) - (\Lambda_i - \xi) \\ &= \Lambda_i - \Lambda_i \subset \Xi(\mathbf{\Lambda}). \end{aligned} \tag{3.11}$$

Thus $\xi + Q^{2M}\Xi(\mathbf{\Lambda}) + Q^{2M}\Xi(\mathbf{\Lambda}) \subset \Lambda_i$. For any overlap $\mathcal{O} = (R, y, S)$ with $R, S \in \mathcal{T}$, $y \in \Xi(\mathbf{\Lambda})$, we can find $M' = M'(\mathcal{O}) \geq 2M$ so that

$$\mathcal{T} \cap (Q^{M'}((y + \mathrm{supp}(R)) \cap \mathrm{supp}(S)) - Q^{M'} y)$$

contains at least one $\xi + Q^{2M}z + T_i$ with some $z \in \Lambda_i - \Lambda_i$, since $Q^{2M}(\Lambda_i - \Lambda_i)$ is relatively dense. Then

$$\xi + Q^{2M}z + T_i \in \omega^{M'}(R).$$

Note that $Q\Xi(\mathbf{\Lambda}) \subset \Xi(\mathbf{\Lambda})$. So

$$\xi + Q^{2M}z + Q^{M'}y \subset \xi + Q^{2M}\Xi(\mathbf{\Lambda}) + Q^{2M}\Xi(\mathbf{\Lambda}) \subset \Lambda_i$$

and

$$\xi + Q^{2M}z + Q^{M'}y + T_i \in \omega^{M'}(S).$$

Thus there is a coincidence after the M'-iteration of the overlap (R, y, S). In particular, since $\Xi(\mathbf{\Lambda})$ is a Meyer set, there are finite equivalence classes of overlaps and so there exists $l \in \mathbb{Z}_+$ such that for each overlap $\mathcal{O}$ in $\mathcal{T}$, $Q^l(\mathcal{O})$ contains a coincidence. Therefore $\mathcal{T}$ admits an overlap coincidence. □

Remark 3.12. Note that the legality of every $\mathbf{\Lambda}$-cluster in a primitive substitution Delone multi-colour set $\mathbf{\Lambda}$ implies the repetitivity of the associated substitution tiling $\mathbf{\Lambda} + \mathcal{A}$ of $\mathbf{\Lambda}$ and vice versa.

Combining the results of Corollary 3.5 and Propositions 3.10 and 3.11, we get the following theorem.

Theorem 3.13. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal and $\mathbf{\Lambda}$ has FLC. Then the following are equivalent:*

(1) *$(X_{\mathbf{\Lambda}}, \mathbb{R}^d, \mu)$ has a pure point dynamical spectrum;*
(2) *$\mathbf{\Lambda}$ admits an algebraic coincidence.* □

4. Algebraic coincidence to inter-model sets

4.1. The Q-topology

Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set. Define

$$L := \langle \Lambda_j \rangle_{j \leq m}$$

the group generated by Λ_j, $j \leq m$, and let

$$\mathcal{K} := \{x \in \mathbb{R}^d : \mathbf{\Lambda} + x = \mathbf{\Lambda}\}$$

be the set of periods of $\mathbf{\Lambda}$. Under the assumption that $\mathbf{\Lambda}$ admits an algebraic coincidence, we introduce a topology on L and find a completion H of the topological group L such that the image of L is a dense subgroup of H. This enables us to construct a cut and project scheme (CPS) such that each point set Λ_i, $i \leq m$, arises from the CPS. In the following lemma we show that the system $\{\alpha + Q^n\Xi(\mathbf{\Lambda}) + \mathcal{K} : n \in \mathbb{Z}_+, \alpha \in L\}$ satisfies the topological properties for the group L to be a topological group [13,9,15].

Lemma 4.1. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set with expansive map Q such that every $\mathbf{\Lambda}$-cluster is legal. Suppose that $\mathbf{\Lambda}$ admits an algebraic coincidence. Then the system $\{\alpha + Q^n\Xi(\mathbf{\Lambda}) + \mathcal{K} : n \in \mathbb{Z}_+, \alpha \in L\}$ serves as a neighbourhood base of the topology on L relative to which L becomes a topological group.*

Proof. From the assumption of an algebraic coincidence there exist $M \in \mathbb{Z}_+$ and $\xi \in \Lambda_i$ such that

$$Q^M\Xi(\mathbf{\Lambda}) \subset \Lambda_i - \xi \quad \text{for some } i \leq m. \tag{4.1}$$

Consider the family $\mathcal{U} = \{Q^n\Xi(\mathbf{\Lambda}) + \mathcal{K} \subset L : n \in \mathbb{Z}_+\}$. We first note that every finite subfamily of $\mathcal{U}$ has a non-empty intersection. Next we will show that $\mathcal{U}$ satisfies the following property : for every $U \in \mathcal{U}$ and $x \in U$, there exist $V \in \mathcal{U}$ and $V' \in \mathcal{U}$ such that $V + V \subset U$ and $x + V' \subset U$. Other properties for a prebase of neighbourhoods of the identity are rather trivial in the Abelian group L. First, note that

$$Q^M \Xi(\mathbf{\Lambda}) + Q^M \Xi(\mathbf{\Lambda}) \subset \Xi(\mathbf{\Lambda}). \tag{4.2}$$

Choose arbitrary $Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K} \in \mathcal{U}$. Then with $V := Q^{M+n}\Xi(\mathbf{\Lambda}) + \mathcal{K}$

$$V + V = Q^{M+n}\Xi(\mathbf{\Lambda}) + \mathcal{K} + Q^{M+n}\Xi(\mathbf{\Lambda}) + \mathcal{K} \subset Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}. \tag{4.3}$$

Second, let $x = Q^n(\alpha - \beta) + k \in Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}$, where $\alpha, \beta \in \Lambda_j$ for some $j \le m$ and $k \in \mathcal{K}$. Since every $\mathbf{\Lambda}$-cluster is legal, there exist $k \in \mathbb{Z}_+$ and $a \in \Lambda_j - \Lambda_j$ such that the cluster $\{\alpha, \beta\}$ satisfies $a + \{\alpha, \beta\} \subset \Phi^k(\xi) \cap \Lambda_j$ for $\xi \in \Lambda_i$ as in (4.1). So we can find $g \in (\Phi^k)_{ji}$ such that $g(\xi) = a + \alpha$. From (4.1), $g(\xi + Q^M \Xi(\mathbf{\Lambda})) \subset \Lambda_j$. So $a + \alpha + Q^{M+k}\Xi(\mathbf{\Lambda}) \subset \Lambda_j$. Since $a + \beta \in \Lambda_j$,

$$\alpha - \beta + Q^{M+k}\Xi(\mathbf{\Lambda}) = a + \alpha - (a + \beta) + Q^{M+k}\Xi(\mathbf{\Lambda}) \subset \Lambda_j - (a + \beta) \subset \Lambda_j - \Lambda_j.$$

Therefore with $V' := Q^{M+k+n}\Xi(\mathbf{\Lambda}) + \mathcal{K}$

$$x + V' = x + Q^{M+k+n}\Xi(\mathbf{\Lambda}) + \mathcal{K} = Q^n(\alpha - \beta) + Q^{M+k+n}\Xi(\mathbf{\Lambda}) + \mathcal{K} \subset Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}.$$

Therefore the system $\{\alpha + Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K} : n \in \mathbb{Z}_+, \alpha \in L\}$ serves as a prebase of neighbourhoods of the topology on L relative to which L becomes a topological group. In fact the system becomes a neighbourhood base for the topology, since for any $n', n \in \mathbb{Z}_+$ with $n' \ge n$,

$$(Q^{n'}\Xi(\mathbf{\Lambda}) + \mathcal{K}) \cap (Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}) = Q^{n'}\Xi(\mathbf{\Lambda}) + \mathcal{K} \in \mathcal{U}. \quad \square$$

We call the topology on L with the neighbourhood base $\{\alpha + Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K} : n \in \mathbb{Z}_+, \alpha \in L\}$ *Q-topology*.

4.2. Construction of a CPS

Let $L' = L/\mathcal{K}$ where L and $\mathcal{K}$ are defined as in Section 4.1. From [9, III. Section 3.4, Section 3.5] and Lemma 4.1, we know that there exists a complete Hausdorff topological group of L', which we denote by H, for which L' is isomorphic to a dense subgroup of the complete group H (see [4,21]). Furthermore there is a uniformly continuous mapping $\psi : L \to H$ which is the composition of the canonical injection of L' into H and the canonical homomorphism of L onto L' for which $\psi(L)$ is dense in H and the mapping ψ from L onto $\psi(L)$ is an open map, the latter with the induced topology of the completion H. One can directly consider H as the Hausdorff completion of L vanishing $\mathcal{K}$.

Theorem 4.2. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set with expansive map Q. Suppose that $\mathbf{\Lambda}$ admits an algebraic coincidence. Then there exists a CPS with the locally compact Abelian group H for an internal space such that for each $j \le m$, $\Lambda_j = \Lambda(V_j)$ where $\overline{V_j}$ is a compact set in H.*

Proof. We claim that for any $n \in \mathbb{Z}_{\ge 0}$, $Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}$ is precompact. The argument is familiar from [4,21], but we provide the argument for the completeness. Note that $\{Q^{n'}\Xi(\mathbf{\Lambda}) + \mathcal{K} : n' \in \mathbb{Z}_{\ge 0}\}$ is a neighbourhood basis for 0. We will show that for any $n' \in \mathbb{Z}_{\ge 0}$, $Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}$ can be covered by some finite translations of $Q^{n'}\Xi(\mathbf{\Lambda}) + \mathcal{K}$. For any $n' \in \mathbb{Z}_{\ge 0}$ with $n' \ge n$, there exists a compact set $C \subset \mathbb{R}^d$ for which $Q^{n'}\Xi(\mathbf{\Lambda}) + C = \mathbb{R}^d$, since $Q^{n'}\Xi(\mathbf{\Lambda})$ is relatively dense. So for any $t \in Q^n \Xi(\mathbf{\Lambda})$, $t = s + c$ where $s \in Q^{n'}\Xi(\mathbf{\Lambda})$ and $c \in C$. Thus

$$c = t - s \in Q^n \Xi(\mathbf{\Lambda}) - Q^{n'}\Xi(\mathbf{\Lambda}) \subset Q^n \Xi(\mathbf{\Lambda}) - Q^n \Xi(\mathbf{\Lambda}).$$

From the assumption of an algebraic coincidence, there exist $M \in \mathbb{Z}_+$ and $\xi \in \Lambda_i$ such that $Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_i - \xi$ for some $i \le m$. So we get

$$Q^{n+M}\Xi(\mathbf{\Lambda}) - Q^{n+M}\Xi(\mathbf{\Lambda}) \subset \Xi(\mathbf{\Lambda}). \tag{4.4}$$

Thus $Q^M c \in \Xi(\mathbf{\Lambda})$. Since $\Xi(\mathbf{\Lambda})$ is discrete, $F := \Xi(\mathbf{\Lambda}) \cap Q^M C$ is finite and $Q^M c \in F$. Thus $t = s + c \in Q^n \Xi(\mathbf{\Lambda}) + Q^{-M} F$ and we obtain that

$$Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K} \subset Q^{n'}\Xi(\mathbf{\Lambda}) + \mathcal{K} + Q^{-M} F.$$

Therefore $Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}$ is precompact. This implies that for each $n \in \mathbb{Z}_{\geq 0}$,

$$\overline{\psi(Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K})} \quad \text{is compact.} \tag{4.5}$$

Define $\widetilde{L} := \{(t, \psi(t)) \in \mathbb{R}^d \times H : t \in L\}$. Applying the same argument as in [4, Sec. 3], we note that $\widetilde{L}$ is a uniformly discrete and relatively dense subgroup in $\mathbb{R}^d \times H$. Then we can construct a cut and project scheme

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L} & \longrightarrow & \psi(L) \\ & & & & \\ t & \longleftarrow & (t, \psi(t)) & \longrightarrow & \psi(t) \end{array} \tag{4.6}$$

where π_1 and π_2 are the canonical projections. It is easy to see that $\pi_1|_{\widetilde{L}}$ is injective and $\pi_2(\widetilde{L})$ is dense in H. We note that for each $j \leq m$, $\Lambda_j = \Lambda_j + \mathcal{K}$. So $\Lambda_j = \Lambda(\psi(\Lambda_j + \mathcal{K}))$ and $\overline{\psi(\Lambda_j + \mathcal{K})}$ is compact in H from (4.5). □

4.3. Existence of model sets

In the proof of the following Lemma 4.3 and Proposition 4.4 we make use of the representability, identifying a primitive substitution Delone multi-colour set $\mathbf{\Lambda}$ with the associated substitution tiling $\mathcal{T} = \mathbf{\Lambda} + \mathcal{A}$ as in Section 2.3.3 where $\mathcal{A}$ is the set of tiles arising from the solution of the adjoint system of equations. Since Φ is primitive, we can assume that the substitution matrix $S(\Phi)$ is positive, replacing Φ by a power of Φ if necessary.

In the next Proposition 4.4 we show that if $\mathbf{\Lambda}$ admits an algebraic coincidence, there exists $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ which is generated from one point and admits an algebraic coincidence at the generating point. This enables us in Proposition 4.5 to show that each point set Γ_i is a model set whose window is open in H in CPS (4.6).

The following lemma is auxiliary to Proposition 4.4.

Lemma 4.3. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set with MFS Φ such that every $\mathbf{\Lambda}$-cluster is legal. Suppose that* $\mathrm{supp}(\eta' + T_j) \subset (\mathrm{supp}(\omega^N(\eta + T_j)))^\circ$ *for some $\eta, \eta' \in \Lambda_j$ and $N \in \mathbb{Z}_+$, and $f \in (\Phi^N)_{jj}$ for which $f(\eta) = \eta'$. Then there exists $\mathbf{\Gamma} = \lim_{n\to\infty}(\Phi^N)^n(y) \in X_{\mathbf{\Lambda}}$ for some fixed point y of f.*

Proof. In this lemma we show how to find a substitution Delone multi-colour set in $X_{\mathbf{\Lambda}}$ which is generated by a fixed point. We can find a fixed point $y \in \mathbb{R}^d$ of f, since f is an affine map with expansive linear part. Note that

$$\omega^N(y + T_j) = \{x + T_i : x \in (\Phi^N)_{ij}(y), i \leq m\}$$

(see [23, Th. 3.7] for the details). So

$$y + T_j \in \omega^N(y + T_j). \tag{4.7}$$

Notice that $\omega^N(y+T_j)$ is translationally equivalent to $\omega^N(\eta+T_j)$ and so the relative location of $y+T_j$ in $\omega^N(y+T_j)$ is same as the relative location of $\eta' + T_j$ in $\omega^N(\eta + T_j)$, since $f(\eta) = \eta'$ and $f(y) = y$. So from the assumption that $\mathrm{supp}(\eta' + T_j) \subset (Q^N(\mathrm{supp}(\eta + T_j)))^\circ$,

$$\mathrm{supp}(y + T_j) \subset (Q^N \mathrm{supp}(y + T_j))^\circ. \tag{4.8}$$

Now we claim that $0 \in (\mathrm{supp}(y + T_j))^\circ$. By (4.8), it is enough to show that

$$0 \in \mathrm{supp}(y + T_j). \tag{4.9}$$

In fact, for any open neighbourhood U of 0, there exists $s \in \mathbb{Z}_+$ such that

$$\mathrm{supp}(y + T_j) \subset Q^s U.$$

From (4.7),

$$Q^{sN} U \cap Q^{sN} \mathrm{supp}(y + T_j) \supset Q^{sN} U \cap \mathrm{supp}(y + T_j) \neq \emptyset.$$

Thus

$$U \cap \mathrm{supp}(y + T_j) \neq \emptyset$$

and so the claim is proved. Let $\mathcal{T}' := \lim_{n\to\infty}(\omega^N)^n(y + T_j)$. Since the generating tile $y + T_j$ contains 0 in the interior, $\mathcal{T}'$ covers $\mathbb{R}^d$ and it is a tiling. Let

$$\mathbf{\Gamma} := \lim_{n\to\infty}(\Phi^N)^n(y).$$

Then $\mathbf{\Gamma}$ is a primitive substitution Delone multi-colour set in $X_{\mathbf{\Lambda}}$ generated from $y \in \Gamma_j$. □

In the following proposition we show that the substitution Delone multi-colour set $\mathbf{\Gamma}$ obtained from Lemma 4.3 admits an algebraic coincidence at the point y, using the assumption of the algebraic coincidence of $\mathbf{\Lambda}$.

Proposition 4.4. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set with expansive map Q and MFS Φ for which every $\mathbf{\Lambda}$-cluster is legal. Suppose that $\mathbf{\Lambda}$ admits an algebraic coincidence. Then there exists $\mathbf{\Gamma} = \lim_{n\to\infty}(\Phi^N)^n(y) \in X_{\mathbf{\Lambda}}$ such that $y + Q^N \Xi(\mathbf{\Lambda}) \subset \Gamma_j$ for some $y \in \Gamma_j$, $j \le m$ and $N \in \mathbb{Z}_+$.*

Proof. By the assumption of algebraic coincidence, there exist $M \in \mathbb{Z}_+$ and $\xi \in \Lambda_i$ such that

$$\xi + Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_i \quad \text{for some } i \le m. \tag{4.10}$$

We again consider the associated substitution tiling $\mathcal{T} := \mathbf{\Lambda} + \mathcal{A}$ of $\mathbf{\Lambda}$, where $\mathcal{A} = \{T_1, \ldots, T_m\}$, and let $A_i = \mathrm{supp}(T_i)$ for $i \le m$.

It is shown in [18] that if $\mathbf{\Lambda}$ is a substitution Delone multi-colour set, then there is a finite multiset (cluster) $\mathbf{P} \subset \mathbf{\Lambda}$ for which $\mathbf{\Lambda} = \lim_{n\to\infty} \Phi^n(\mathbf{P})$. So we can find $\eta \in \Lambda_j$ for some $j \le m$ and $M' \in \mathbb{Z}_+$ with $M' \ge M$ such that $\eta + T_j$ is fixed under ω and $\omega^{M'}(\eta + T_j)$ contains $\xi + T_i$. By the primitivity, we can choose a j-type tile $\eta' + T_j$ in the interior of $\omega^K(\xi + T_i)$ with some $K \in \mathbb{Z}_+$. So

$$\mathrm{supp}(\eta' + T_j) \subset (\mathrm{supp}(\omega^K(\xi + T_i)))^\circ \subset (\mathrm{supp}(\omega^{M'+K}(\eta + T_j)))^\circ \subset (\mathrm{supp}(\omega^N(\eta + T_j)))^\circ,$$

where $N = 2(M' + K)$. From Lemma 4.3, there exists

$$\mathbf{\Gamma} = \lim_{n\to\infty}(\Phi^N)^n(y) \in X_{\mathbf{\Lambda}} \tag{4.11}$$

for some fixed point y of f where $f(\eta) = \eta'$ and $f \in (\Phi^N)_{jj}$. Let $\mathcal{T}' = \mathbf{\Gamma} + \mathcal{A}$.

We are going to show that $\mathbf{\Gamma}$ admits an algebraic coincidence at y using the algebraic coincidence at ξ for $\mathbf{\Lambda}$ and using the repetitivity of $\mathcal{T}$. Note that $\eta' \in \Phi^K(\xi)$, which means that there exists $h \in (\Phi^K)_{ji}$ such that $\eta' = h(\xi)$. Then applying MFS Φ to the inclusion by (4.10), we get $h(\xi + Q^M \Xi(\mathbf{\Lambda})) \subset h(\Lambda_i) \subset \Lambda_j$. Thus $\eta' + Q^{M+K}\Xi(\mathbf{\Lambda}) \subset \Lambda_j$. Then (4.10) we get

$$\eta' + Q^{M+K}\Xi(\mathbf{\Lambda}) \subset \Lambda_j.$$

Since $Q\Xi(\mathbf{\Lambda}) \subset \Xi(\mathbf{\Lambda})$, we have $\eta' + Q^{M'+K}\Xi(\mathbf{\Lambda}) \subset \Lambda_j$. Thus

$$Q^{M'+K}\Xi(\mathbf{\Lambda}) - Q^{M'+K}\Xi(\mathbf{\Lambda}) \subset \Xi(\mathbf{\Lambda}).$$

Let $N = 2(M' + K)$. So

$$\eta' + Q^N \Xi(\mathbf{\Lambda}) - Q^N \Xi(\mathbf{\Lambda}) \subset \Lambda_j. \tag{4.12}$$

Note that $\Xi(\mathcal{T}')$ is also a Meyer set. So as in (3.2) we can find $a_1, \ldots, a_S \in \Xi(\mathcal{T}')$ such that for any $a \in \Xi(\mathcal{T}')$,

$$\mathcal{T}' \cap (y + A_j - a) + a = \mathcal{T}' \cap (y + A_j - a_s) + a_s \quad \text{for some } s \le S, \tag{4.13}$$

where $y \in \Gamma_j$ as in (4.11). There exists $p \in \mathbb{Z}_+$ that $(\omega^N)^p(y + T_j)$ contains the patches of $\mathcal{T}' \cap (y + A_j - a_1), \ldots, \mathcal{T}' \cap (y + A_j - a_S)$ from (4.11). Since $\mathcal{T}$ is repetitive, there is $r \in \mathbb{R}^d$ such that

$$\mathcal{T}' \cap (y + A_j - a_s) + r \subset (\omega^N)^p(y + T_j) + r \subset \mathcal{T} \quad \text{for all } s \le S. \tag{4.14}$$

Note that $y + r + T_j, \eta + T_j \in \mathcal{T}$. Take any $s \le S$. Since $y + r - \eta \in \Xi(\mathcal{T})$, by (4.12) we obtain

$$\eta' + T_j + Q^N(y + r - \eta) - Q^N a_s \in \mathcal{T}. \tag{4.15}$$

Now we want to show that the tile on the left-hand side of (4.15) is in a patch $\omega^N(\mathcal{T}' \cap (y + A_j - a_s) + r)$. Since $\eta' + T_j \in \omega^N(\eta + T_j)$,

$$\eta' + T_j + Q^N(y + r - \eta) - Q^N a_s \in \omega^N(y + T_j + r - a_s).$$

Moreover

$$\mathrm{supp}(\omega^N(y + T_j + r - a_s)) \subset \mathrm{supp}(\omega^N(\mathcal{T}' \cap (y + A_j - a_s) + r))$$

and

$$\omega^N((\mathcal{T}' \cap (y + A_j - a_s)) + r) \subset \mathcal{T}$$

from (4.14). So we can see that

$$\eta' + T_j + Q^N(y + r - \eta) - Q^N a_s \in \omega^N(\mathcal{T}' \cap (y + A_j - a_s) + r). \tag{4.16}$$

Let $f : x \mapsto Q^N x + e$, where $e \in \mathbb{R}^d$. Then we have the identities $\eta' = Q^N \eta + e$ and $y = Q^N y + e$. Applying these identities to (4.16), we get, for all $s \le S$,

$$y + T_j \in \omega^N(\mathcal{T}' \cap (y + A_j - a_s) + a_s).$$

Hence through (4.13), for any arbitrary $a \in \Xi(\mathbf{\Lambda})$,

$$y + T_j \in \omega^N(\mathcal{T}' \cap (y + A_j - a) + a).$$

Thus $y + T_j - Q^N a \in \omega^N(\mathcal{T}' \cap (y + A_j - a)) \subset \mathcal{T}'$. Since a is arbitrary in $\Xi(\mathbf{\Gamma})$, $y + Q^N \Xi(\mathbf{\Gamma}) \subset \Gamma_j$. □

Proposition 4.5. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set. Suppose that if $\mathbf{\Lambda} = \lim_{n\to\infty} \Phi^n(y)$ where $y + Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_j$ and $y \in \Lambda_j$ for some $M \in \mathbb{Z}_+$ and $j \le m$. Then for each $i \le m$, $\Lambda_i = \Lambda(U_i)$ in CPS (4.6) where U_i is an open set and $\overline{U_i}$ is compact in the internal space H, i.e. Λ_i is a model set with an open window.*

Proof. For each $i \le m$ and $z \in \Lambda_i$, there exists $n \in \mathbb{Z}_+$ such that

$$z = Q^n y + e \quad \text{for some } e \in \mathbb{R}^d,$$

where $f : x \mapsto Q^n x + e$ and $f \in (\Phi^n)_{ij}$. From $y + Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_j$, $z + Q^{n+M} \Xi(\mathbf{\Lambda}) \subset \Lambda_i$. Moreover, $z + Q^{n+M} \Xi(\mathbf{\Lambda}) + \mathcal{K} \subset \Lambda_i$, since $\Lambda_i = \Lambda_i + \mathcal{K}$. Thus

$$\Lambda_i = \bigcup_{z \in \Lambda_i} (z + Q^{M_z} \Xi(\mathbf{\Lambda}) + \mathcal{K}),$$

where M_z depends on z. Since ψ is an open map from L onto $\psi(L)$, where the latter is with the induced topology of the completion H, for each $z + Q^{M_z} \Xi(\mathbf{\Lambda}) + \mathcal{K}$ there exists an open set U_z in H such that

$$\psi(z + Q^{M_z} \Xi(\mathbf{\Lambda}) + \mathcal{K}) = \psi(L) \cap U_z.$$

Since $\mathrm{Ker}(\psi) = K$ and $\Lambda_i = \Lambda_i + K$, $\Lambda_i = \psi^{-1}(\psi(L) \cap U_i) = \Lambda(U_i)$ where $U_i = \bigcup_{z \in \Lambda_i} U_z$. Furthermore $\overline{\psi(\Lambda_i)} = \overline{U_i}$ by the denseness of $\psi(L)$ in H and $\overline{U_i}$ is compact by (4.5). □

4.4. Two equivalent topologies on L

In this subsection we introduce another topology on L which becomes equivalent to Q-topology under the assumption of algebraic coincidence. Theorem 4.12 shows a sufficient condition to get inter-model set in general setting but the CPS in the theorem is constructed based upon on the new topology. The equivalence of the two topologies gives us the equivalence of the two CPSs. We make use of Theorem 4.12 to get connection to inter-model multi-colour sets for substitution Delone multi-colour set.

Let $\{F_n\}_{n\in\mathbb{Z}_+}$ be a van Hove sequence and let $\mathbf{\Lambda}'$, $\mathbf{\Lambda}''$ be two Delone m-multi-colour sets in $\mathbb{R}^d$. We define

$$\rho(\mathbf{\Lambda}', \mathbf{\Lambda}'') := \lim_{n\to\infty} \sup \frac{\sum_{i=1}^{m} \sharp((\Lambda_i' \triangle \Lambda_i'') \cap F_n)}{\mathrm{Vol}(F_n)}. \tag{4.17}$$

Here $\triangle$ is the symmetric difference operator. Let $P_\epsilon = \{x \in L : \rho(x+\mathbf{\Lambda}, \mathbf{\Lambda}) < \epsilon\}$ for each $\epsilon > 0$. From Theorem 3.13 and [23, Lemma A.9], if $\mathbf{\Lambda}$ admits an algebraic coincidence, then, for any $\epsilon > 0$, P_ϵ is relatively dense. In this case the system $\{\alpha + P_\epsilon : \epsilon > 0, \alpha \in L\}$ serves as a neighbourhood base of the topology on L relative to which L becomes a topological group. We name P_ϵ-topology for this topology on L and denote the space L with P_ϵ-topology by L_P (see [4] for P_ϵ-topology under the name of autocorrelation topology).

Let L_Q be the space L with Q-topology. In the following two propositions we show that L_Q is topologically isomorphic to L_P.

Proposition 4.6. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal. If $\mathbf{\Lambda}$ admits an algebraic coincidence, then the mapping $\iota : x \mapsto x$ from L_Q onto L_P is uniformly continuous.*

Proof. It is enough to show that for each $\epsilon > 0$, there exists $n \in \mathbb{Z}_+$ such that $Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K} \subset P_\epsilon$. Let $\mathcal{T} = \mathbf{\Lambda} + \mathcal{A}$ be the associated substitution tiling of $\mathbf{\Lambda}$. The assumption of algebraic coincidence gives overlap coincidence and, from [23, Lemma A.9], there exist $r \in (0, 1)$ and $C > 0$ such that for any $x \in \Xi(\mathcal{T})$

$$1 - \mathrm{dens}(\mathcal{T} \cap (Q^n x + \mathcal{T})) \le C r^n.$$

Since

$$\mathrm{dens}(\mathcal{T} \cap (Q^n x + \mathcal{T})) = \sum_{i=1}^{m} \mathrm{freq}(\{T_i, Q^n x + T_i\}, \mathcal{T}) \cdot \mathrm{Vol}(A_i)$$

and

$$\mathrm{freq}(\{T_i, Q^n x + T_i\}, \mathcal{T}) = \mathrm{dens}(\Lambda_i \cap (Q^n x + \Lambda_i)),$$

we get

$$\begin{aligned} 1 - \mathrm{dens}(\mathcal{T} \cap (Q^n x + \mathcal{T})) &= 1 - \sum_{i=1}^{m} \mathrm{dens}(\Lambda_i \cap (Q^n x + \Lambda_i)) \cdot \mathrm{Vol}(A_i) \\ &= \sum_{i=1}^{m} \mathrm{dens}(\Lambda_i) \cdot \mathrm{Vol}(A_i) - \sum_{i=1}^{m} \mathrm{dens}(\Lambda_i \cap (Q^n x + \Lambda_i)) \cdot \mathrm{Vol}(A_i) \\ &= \sum_{i=1}^{m} \frac{1}{2}(\mathrm{dens}(\Lambda_i \triangle (Q^n x + \Lambda_i))) \cdot \mathrm{Vol}(A_i). \end{aligned}$$

Let $V_0 = \min\{\mathrm{Vol}(A_i) : i \le m\}$. Then

$$V_0 \cdot \sum_{i=1}^{m} \frac{1}{2}(\mathrm{dens}(\Lambda_i \triangle (Q^n x + \Lambda_i))) \le C r^n.$$

So for all $x \in \Xi(\mathbf{\Lambda})$,

$$\mathrm{dens}(\mathbf{\Lambda} \triangle (Q^n x + \mathbf{\Lambda})) \le C' r^n \quad \text{where } C' = \frac{2C}{V_0} > 0.$$

Then $Q^n x \in P_{C' r^n}$ and thus for any $\epsilon > 0$, we can find $n \in \mathbb{Z}_+$ satisfying $C' r^n < \epsilon$ so that

$$Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K} \subset P_\epsilon. \quad \square$$

Proposition 4.7. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal. If $\mathbf{\Lambda}$ admits an algebraic coincidence, then the mapping $\iota^{-1} : x \mapsto x$ from L_P onto L_Q is uniformly continuous.*

Proof. It is enough to show that for any $n \in \mathbb{Z}_+$ there exists $\epsilon > 0$ such that $P_\epsilon \subset Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}$. For the associated substitution tiling $\mathcal{T} = \mathbf{\Lambda} + \mathcal{A}$ of $\mathbf{\Lambda}$. Since $\Xi(\mathcal{T})$ is a Meyer set from Lemma 3.9, we can find small $\zeta > 0$ such that $\mathcal{T} \cap (r + t + \mathcal{T}) = \emptyset$ for all $t \in \Xi(\mathcal{T})$ and $r \in B_\zeta(0) \setminus \{0\}$. From [32, Lemma 3.5 and Th. 2.14], for any $0 < \epsilon \le \frac{\zeta}{\|Q^n\|}$ there exists $\delta_\epsilon > 0$ such that if $d(\mathcal{T} - x, \mathcal{T} - (x + t)) < \delta_\epsilon$ for some $x \in \mathbb{R}^d$ and $t \in \Xi(\mathcal{T})$, then there exist $g_j, g'_j \in \mathcal{K}$ for $1 \le j \le n$, such that

$$d\left(\mathcal{T} - Q^{-n}x - \sum_{j=1}^{n} Q^{-n+j-1} g_j, \mathcal{T} - Q^{-n}(x+t) - \sum_{j=1}^{n} Q^{-n+j-1} g'_j\right) < \epsilon.$$

By the metric d on $X_{\mathcal{T}}$ of (2.4), there exists $h \in B_\epsilon(0)$ such that

$$\mathcal{T} - Q^{-n}x - \sum_{j=1}^{n} Q^{-n+j-1} g_j - h \quad \text{agrees with } \mathcal{T} - Q^{-n}(x+t) - \sum_{j=1}^{n} Q^{-n+j-1} g'_j \text{ on } B_{1/\epsilon}(0).$$

So

$$\mathcal{T} - x - \sum_{j=1}^{n} Q^{j-1} g_j - Q^n h \quad \text{agrees with } \mathcal{T} - (x+t) - \sum_{j=1}^{n} Q^{j-1} g'_j \quad \text{on } Q^n B_{1/\epsilon}(0).$$

Note that

$$t - \sum_{j=1}^{n} Q^{j-1} g_j + \sum_{j=1}^{n} Q^{j-1} g'_j \in \Xi(\mathcal{T}),$$

since $Q\mathcal{K} \subset \mathcal{K}$ and $\Xi(\mathcal{T}) + \mathcal{K} = \Xi(\mathcal{T})$. Since $\|Q^n\|\epsilon \le \zeta$ and the choice of ξ, $Q^n h = 0$. Thus $h = 0$. So

$$Q^{-n}t - \sum_{j=1}^{n} Q^{-n+j-1} g_j + \sum_{j=1}^{n} Q^{-n+j-1} g'_j \in \Xi(\mathcal{T}).$$

Thus

$$t = Q^n z + w, \quad \text{where } z \in \Xi(\mathcal{T}) \text{ and } w = \sum_{j=1}^{n} Q^{j-1}(g_j - g'_j) \in \mathcal{K},$$

and hence $t \in Q^n \Xi(\mathcal{T}) + \mathcal{K}$.

If $t \in P_\epsilon$, $\rho(t + \mathbf{\Lambda}, \mathbf{\Lambda}) < \epsilon$. This means that for small $\epsilon > 0$ there is a big area of overlaps in $\mathbb{R}^d$ between $t + \mathcal{T}$ and $\mathcal{T}$ so that $d(\mathcal{T} + x, \mathcal{T} + x - t)$ is small for some $x \in \mathbb{R}^d$. So we can choose small $\epsilon > 0$ so that for any $t \in P_\epsilon$, $d(\mathcal{T} + x, \mathcal{T} + x - t) < \delta_\epsilon$ for some $x \in \mathbb{R}^d$ by the definition of P_ϵ. Then $t \in Q^n \Xi(\mathcal{T}) + \mathcal{K}$. Hence $P_\epsilon \subset Q^n \Xi(\mathbf{\Lambda}) + \mathcal{K}$. □

Remark 4.8. From Propositions 4.6 and 4.7, L_P is topologically isomorphic to L_Q. Thus the completion of L_P is topologically isomorphic to the completion H of L_Q. We will identify the former with H. Thus $\phi := \psi \cdot \iota^{-1} : L_P \to H$ is uniformly continuous, $\phi(L_P)$ is dense in H, and the mapping ϕ from L_P onto $\phi(L_P)$ is an open map, the latter with the induced topology of the completion H. Therefore we can consider the CPS (4.6) with an internal space H which is a completion of L_P. Note that since $\mathbf{\Lambda}$ is repetitive, $\bigcap_{\epsilon>0} P_\epsilon = \mathcal{K}$ and $\mathcal{K} = \overline{\{0\}}$ in L_Q.

4.5. Inter-model sets

4.5.1. A continuous map between two dynamical hulls

In this subsection we show that for a primitive substitution Delone multi-colour set $\mathbf{\Lambda}$ with pure point spectrum, there exists a continuous map from $X_{\mathbf{\Lambda}}$ to $\mathbb{A}(\mathbf{\Lambda})$. This continuous map was first introduced in [3].

Let us define an *autocorrelation group* $\mathbb{A}(\mathbf{\Lambda})$. Let $\widetilde{\mathcal{D}}$ be the set of all Delone m-multi-colour sets in $\mathbb{R}^d$. Define an equivalence relation on $\widetilde{\mathcal{D}}$ by $\mathbf{\Lambda}' \equiv \mathbf{\Lambda}'' \Leftrightarrow \rho(\mathbf{\Lambda}', \mathbf{\Lambda}'') = 0$. Let $\mathcal{D} := \widetilde{\mathcal{D}}/\equiv$ and let ρ also denote the resulting $\mathbb{R}^d$-invariant metric on $\mathcal{D}$. Now we define a new uniformity on $\mathcal{D}$, which mixes the autocorrelation topology with the standard topology of $\mathbb{R}^d$ using the sets

$$U(V, \epsilon) = \{(\mathbf{\Lambda}', \mathbf{\Lambda}'') \in \mathcal{D} \times \mathcal{D} : \rho(-v + \mathbf{\Lambda}', \mathbf{\Lambda}'') < \epsilon \text{ for some } v \in V\}$$

where $\epsilon > 0$ and V is a neighbourhood of 0. Then $\mathcal{D}$ is a complete space [3] and its elements can be identified as Delone multi-colour sets in $\mathbb{R}^d$ up to density 0 changes. Notice that the topology induced by this uniformity is not same with the topology (P_ϵ-topology) induced by the metric ρ. We define $\mathbb{A}(\mathbf{\Lambda})$ as the closure of the orbit $\mathbb{R}^d + \mathbf{\Lambda}$ with the new uniformity in $\mathcal{D}$ (see [3,21,27] for more about $\mathbb{A}(\mathbf{\Lambda})$).

The following theorem is proved in [3] in Delone sets with one colour. The argument can be extended into Delone multi-colour sets without difficulty.

Theorem 4.9. *Let $\mathbf{\Lambda}$ be a Delone multi-colour set in $\mathbb{R}^d$ with UCF such that $\Xi(\mathbf{\Lambda})$ is a Meyer set. If the dynamical system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ has pure point spectrum with continuous eigenfunctions, then there exists a continuous $\mathbb{R}^d$-map*

$$\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda}),$$

in which $\beta : \mathbf{\Gamma} \mapsto \mathbf{\Gamma} \bmod \equiv$. $\square$

In substitution Delone multi-colour sets the condition of continuous eigenfunctions is already implicit:

Theorem 4.10 (*[32, Th. 2.13]*). *Suppose that $\mathbf{\Lambda}$ is a primitive substitution Delone multi-colour set with FLC such that every $\mathbf{\Lambda}$-cluster is legal. Then every measurable eigenfunction for the system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ coincides with a continuous function μ-a.e.*

Since a primitive substitution Delone multi-colour set with FLC has UCF (see [23]), we combine Theorems 4.9, 4.10 and 3.4 and get the following corollary.

Corollary 4.11. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set with FLC such that every $\mathbf{\Lambda}$-cluster is legal. If the dynamical system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ has pure point spectrum, then there exists a continuous $\mathbb{R}^d$-map*

$$\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda}),$$

in which $\beta : \mathbf{\Gamma} \mapsto \mathbf{\Gamma} \bmod \equiv$. $\square$

4.5.2. Algebraic coincidence to inter-model sets

In this subsection we show that if a substitution Delone multi-colour set admits an algebraic coincidence then it is an inter-model multi-colour set.

A continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$ is called a torus parametrization on $X_{\mathbf{\Lambda}}$.[3] An element $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ is *non-singular* for this parametrization if $\beta^{-1}(\beta(\{\mathbf{\Gamma}\})) = \{\mathbf{\Gamma}\}$. The set of non-singular elements of $X_{\mathbf{\Lambda}}$ is invariant under the translation action of $\mathbb{R}^d$.

The result of the following theorem is based on a CPS, taking the completion of L_P as an internal space (see [21]). Since we have shown that the completion of L_P is topologically isomorphic to the completion of L_Q, we can use the CPS (4.6) in Theorem 4.12.

Notice that $\mathbb{A}(\mathbf{\Lambda})$ is isomorphic to a torus $(\mathbb{R}^d \times H)/\widetilde{L}$ by [21, Prop. 3.2].

Theorem 4.12 (*[21, Prop. 4.6]*). *Let $\mathbf{\Lambda}$ be a multi-colour set in $\mathbb{R}^d$ with repetitivity. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$ and $\Lambda(V_i{}^\circ) \subset \Lambda_i \subset \Lambda(\overline{V_i})$ where $\overline{V_i}$ is compact, $V_i{}^\circ \neq \emptyset$, and ∂V_i has empty interior for each $i \le m$ with respect to CPS (4.6). Then there exists a non-singular element $\mathbf{\Lambda}'$ in $X_{\mathbf{\Lambda}}$ such that $\Lambda'_i = \Lambda(W_i)$ where W_i is compact and $W_i = \overline{W_i{}^\circ}$ for each $i \le m$ with respect to the same CPS, and so for each $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ there exists $(-s, -h) \in \mathbb{R}^d \times H$ so that*

$$-s + \Lambda(h + W_i{}^\circ) \subset \Gamma_i \subset -s + \Lambda(h + W_i) \quad \textit{for each } i \le m.$$

In other words, every $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ is an inter-model multi-colour set.

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[3] The terminology of a torus parametrization arises from the model set cases first studied where (in the set-up that we have here) $\mathbb{A}(\mathbf{\Lambda})$ would have been a torus.

Theorem 4.13. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal. Suppose that $\mathbf{\Lambda}$ admits an algebraic coincidence. Then for each $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ there exists $(-s, -h) \in \mathbb{R}^d \times H$ satisfying*

$$-s + \Lambda(h + {W_i}^{\circ}) \subset \Gamma_i \subset -s + \Lambda(h + W_i) \quad \text{for each } i \le m,$$

where W_i is compact and $W_i = \overline{{W_i}^{\circ}} \neq \emptyset$, with respect to the CPS (4.6). In other words, every $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ is an inter-model multi-colour set. In particular, $\mathbf{\Lambda}$ is an inter-model multi-colour set.

Proof. From Propositions 4.4 and 4.5, there exists $\mathbf{\Lambda}' \in X_{\mathbf{\Lambda}}$ for which each $\Lambda'_i = \Lambda(V_i)$, where $V_i \neq 0$ is an open set and $\overline{V_i}$ is compact in H. Here we note that the boundary ∂V_i of the open set V_i has empty interior. Note that algebraic coincidence implies pure point dynamical spectrum by Theorem 3.13. Applying Corollary 4.11 and Theorem 4.12 to $\mathbf{\Lambda}'$, we complete the proof of the theorem. □

5. Inter-model sets to algebraic coincidence

We will show that if a substitution Delone multi-colour set $\mathbf{\Lambda}$ is an inter-model multi-colour set then $\mathbf{\Lambda}$ admits an algebraic coincidence.

For each compact set $K \subset \mathbb{R}^d$, we define

$$T_K(\mathbf{\Lambda}) := \{t \in L : t + (\mathbf{\Lambda} \cap K) = \mathbf{\Lambda} \cap (t + K)\}.$$

We prove the following auxiliary lemma for Theorem 5.2.

Lemma 5.1. *Let $\mathbf{\Lambda}$ be a Delone multi-colour set in $\mathbb{R}^d$. Suppose that for each $i \le m$, $\Lambda({W_i}^{\circ}) \subset \Lambda_i \subset \Lambda(W_i)$ for some compact set $W_i \neq \emptyset$ with $W_i = \overline{{W_i}^{\circ}}$, in some CPS. Then $T_F(\mathbf{\Lambda}) - T_F(\mathbf{\Lambda}) \subset -\xi + \Lambda_j$ for some compact set F and $\xi \in \Lambda({W_i}^{\circ})$.*

Proof. For any $t \in T_K(\mathbf{\Lambda})$, $t + (\mathbf{\Lambda} \cap K) \subset \mathbf{\Lambda} \cap (t + K)$ and $t + (\mathbf{\Lambda} \cap K) \supset \mathbf{\Lambda} \cap (t + K)$. So for each $i \le m$,

$$\psi(t) + \psi(s) \in W_i \quad \forall s \in \Lambda_i \cap K \quad \text{and}$$
$$\psi(t) + \psi(s) \notin {W_i}^{\circ} \quad \forall s \in (L \setminus \Lambda_i) \cap K.$$

Let

$$W_{i,K} := \bigcap \{-\psi(s) + W_i : s \in \Lambda_i \cap K\} \setminus \bigcup \{-\psi(s) + {W_i}^{\circ} : s \in (L \setminus \Lambda_i) \cap K\},$$

for each $i \le m$. Then we can say that

$$T_K(\mathbf{\Lambda}) \subset \Lambda\left(\bigcap_{i \le m} W_{i,K}\right).$$

Fix any $j \le m$. Since ${W_j}^{\circ} \neq \emptyset$, we can find ξ such that $\xi \in \Lambda({W_j}^{\circ}) \subset \Lambda_j$. Since $-\psi(\xi) + {W_j}^{\circ}$ contains a neighbourhood of 0 and H is a locally compact Abelian group, there is a neighbourhood U of 0 in H such that $U - U \subset -\psi(\xi) + {W_j}^{\circ}$. Let

$$I = \{t \in H : t + W_i = W_i \;\text{ for all } i \le m\}.$$

Since $W_i = \overline{{W_i}^{\circ}}$ for all $i \le m$,

$$\{t \in H : t + {W_i}^{\circ} = {W_i}^{\circ} \;\text{ for all } i \le m\} = I.$$

So

$$\begin{aligned}(U + I) - (U + I) &= U - U + I \\ &\subset -\psi(\xi) + {W_j}^{\circ} + I \\ &= -\psi(\xi) + {W_j}^{\circ}.\end{aligned}$$

Note that $U + I$ is a neighbourhood of 0 in H.

We claim that

$$\bigcap\{-\psi(s) + W_i : s \in \Lambda_i, i \le m\} = I.$$

Notice first that

$$0 \in \bigcap\{-\psi(s) + W_i : s \in \Lambda_i, i \le m\} \ne \emptyset.$$

For any $c \in \bigcap\{-\psi(s) + W_i : s \in \Lambda_i, i \le m\}$, $\psi(\Lambda_i) \subset -c + W_i$ for all $i \le m$. So $\overline{\psi(\Lambda_i)} = W_i \subset -c + W_i$ for all $i \le m$. In fact, $W_i = -c + W_i$ for all $i \le m$, since $W_i - W_i$ is compact (see [21, Prop. 5.2] for the detailed proof). Thus $\bigcap\{-\psi(s) + W_i : s \in \Lambda_i, i \le m\} \subset I$. On the other hand, for any $c' \in I$, $c' + W_i = W_i$ for all $i \le m$, and so

$$-\psi(s) + c' + W_i = -\psi(s) + W_i \quad \text{for all } s \in \Lambda_i \text{ and } i \le m.$$

Since $0 \in \bigcap\{-\psi(s) + W_i : s \in \Lambda_i, i \le m\}$,

$$c' \in -\psi(s) + W_i \quad \text{for all } s \in \Lambda_i \text{ and } i \le m.$$

This shows that $c' \in \bigcap\{-\psi(s) + W_i : s \in \Lambda_i, i \le m\}$. Therefore the claim is proved.

So now we have $\bigcap\{(-\psi(s) + W_i) \setminus (U + I) : s \in \Lambda_i, i \le m\} = \emptyset$. Since each $(-\psi(s) + W_i) \setminus (U + I)$ is compact, by the finite intersection property for compact sets there is a finite set $F \subset L$ such that $F \subset \bigcup_{i \le m} \Lambda_i$ and

$$\bigcap\{(-\psi(s) + W_i) \setminus (U + I) : s \in \Lambda_i \cap F, i \le m\} = \emptyset.$$

Thus $U + I \supset \bigcap\{-\psi(s) + W_i : s \in \Lambda_i \cap F,\ i \le m\} \supset \bigcap_{i \le m} W_{i,F}$. Then for the compact set $F \subset \mathbb{R}^d$,

$$\begin{aligned} T_F(\mathbf{\Lambda}) - T_F(\mathbf{\Lambda}) &\subset \Lambda\left(\bigcap_{i \le m} W_{i,F}\right) - \Lambda\left(\bigcap_{i \le m} W_{i,F}\right) \\ &\subset \Lambda((U + I) - (U + I)) \\ &\subset \Lambda(-\psi(\xi) + W_j{}^\circ) \\ &\subset -\xi + \Lambda_j. \quad \square \end{aligned} \tag{5.1}$$

Theorem 5.2. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal. Suppose that $\mathbf{\Lambda}$ is an inter-model multi-colour set. Then $\mathbf{\Lambda}$ admits an algebraic coincidence.*

Proof. From the assumption there is a following cut and project scheme:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H, \\ & & \cup & & \\ & & \widetilde{L} & & \end{array} \tag{5.2}$$

where H is a locally compact Abelian group, $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times H$, π_1 and π_2 are canonical projections, $\pi_1|_{\widetilde{L}}$ is one-to-one, and $\pi_2(\widetilde{L})$ is dense in H. Let $L = \pi_1(\widetilde{L})$. We define $\psi : L \to H$ by $\psi(x) = \pi_2(\pi_1^{-1}(x))$. Then $s + \Lambda(W_i{}^\circ) \subset \Lambda_i \subset s + \Lambda(W_i)$ for some $s \in \mathbb{R}^d$ and non-empty compact set W_i with $W_i = \overline{W_i{}^\circ}$ for each $i \le m$ with respect to the CPS (5.2). We can assume that $\Lambda(W_i{}^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ without loss of generality.

From Lemma 5.1, we have

$$T_F(\mathbf{\Lambda}) - T_F(\mathbf{\Lambda}) \subset -\xi + \Lambda_j \tag{5.3}$$

for some compact set F and $\xi \in \Lambda(W_i{}^\circ)$. Let $K = F + \bigcup_{i \le m}(\mathrm{supp}(T_i))$. Since every $\mathbf{\Lambda}$-cluster is legal, there exist $\alpha \in \Lambda_k$ for some $k \le m$ and $N \in \mathbb{Z}_+$ satisfying $\mathbf{\Lambda} \cap K \subset z + \Phi^N(\alpha)$ for some $z \in \mathbb{R}^d$. Note that for any $\beta \in \Lambda_k$,

$$z + Q^N(\beta - \alpha) + \Phi^N(\alpha) = z + \Phi^N(\beta).$$

Thus $Q^N(\beta - \alpha) + (\mathbf{\Lambda} \cap K) \subset z + \Phi^N(\beta)$ and so

$$-z + Q^N(\beta - \alpha) + (\mathbf{\Lambda} \cap K) \subset \mathbf{\Lambda}.$$

We note further that

$$-z + Q^N(\beta - \alpha) + (\mathbf{\Lambda} \cap K) \subset \mathbf{\Lambda} \cap (-z + Q^N(\beta - \alpha) + K).$$

By the choice of K and the fact that $\mathbf{\Lambda} + \mathcal{A}$ is a tiling,

$$-z + Q^N(\beta - \alpha) + (\mathbf{\Lambda} \cap F) = \mathbf{\Lambda} \cap (-z + Q^N(\beta - \alpha) + F).$$

So $-z + Q^N(\beta - \alpha) \in T_F(\mathbf{\Lambda})$. This shows that $-z + Q^N(\Lambda_k - \alpha) \in T_F(\mathbf{\Lambda})$. Thus $Q^N(\Lambda_k - \Lambda_k) \subset T_F(\mathbf{\Lambda}) - T_F(\mathbf{\Lambda})$. By (5.3) and the fact that $Q^{N+l}\Xi(\mathbf{\Lambda}) \subset Q^N(\Lambda_k - \Lambda_k)$ for some $l \in \mathbb{Z}_+$ by the primitivity of the substitution, there exist $M \in \mathbb{Z}_+$ and $\xi \in \Lambda_j$ such that $Q^M \Xi(\mathbf{\Lambda}) \subset \Lambda_j - \xi$. This completes the proof. □

The following theorem states the main result of this paper.

Theorem 5.3. *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set such that every $\mathbf{\Lambda}$-cluster is legal and $\mathbf{\Lambda}$ has FLC. Then the following are equivalent:*

(1) *$\mathbf{\Lambda}$ has pure point dynamical spectrum;*
(2) *$\mathbf{\Lambda}$ admits an algebraic coincidence;*
(3) *$\mathbf{\Lambda}$ is an inter-model multi-colour set.*

Proof. The proof goes as follows:

(1) ⇔ (2) by Theorem 3.13.
(2) ⇔ (3) by Theorems 4.13 and 5.2. □

Any tiling $\mathcal{T}$ can be converted into a Delone multiset by simply choosing a point for each tile so that the chosen points for tiles of the same type are in the same relative position in the tiles. So we give a corresponding result of Theorem 5.3 on substitution tilings.

The following lemma is taken from Lecture Note of Boris Solomyak. We provide the proof here, since there is no direct reference for it. One can see similar arguments in [18,33,23].

Lemma 5.4. *Let $\mathcal{T}$ be a repetitive fixed point of a primitive substitution such that $\mathcal{T} = \bigcup_{j=1}^m (T_j + \Lambda_j)$. Then $\mathbf{\Lambda}_{\mathcal{T}} := (\Lambda_i)_{i \le m}$ is a primitive substitution Delone multiset and every $\mathbf{\Lambda}_{\mathcal{T}}$-cluster is legal.*

Proof. Let ω be the corresponding tile-substitution for $\mathcal{T}$. Then

$$\mathcal{T} = \bigcup_{j=1}^m (\omega(T_j) + Q\Lambda_j) = \bigcup_{j=1}^m \left(\bigcup_{i=1}^m (T_i + \mathcal{D}_{ij}) + Q\Lambda_j \right) = \bigcup_{i=1}^m \left(T_i + \bigcup_{j=1}^m (Q\Lambda_j + \mathcal{D}_{ij}) \right).$$

Thus

$$\Lambda_i = \bigcup_{j=1}^m (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m.$$

Every $\mathbf{\Lambda}_{\mathcal{T}}$-cluster is legal from Remark 3.12. □

Theorem 5.5. *Let $\mathcal{T}$ be a repetitive fixed point of a primitive substitution with FLC. Then the following are equivalent:*

(1) *$\mathcal{T}$ has pure point dynamical spectrum;*
(2) *$\mathcal{T}$ admits an overlap coincidence;*
(3) *$\mathbf{\Lambda}_{\mathcal{T}}$ is an inter-model multi-colour set.*

Proof. The proof goes as follows:

(1) ⇔ (2) by Corollary 3.5.
(2) ⇔ (3) by Propositions 3.10, 3.11 and Theorem 5.3. □

6. Further study

When the legality in Theorem 5.3 is dropped, finding the corresponding substitution tilings is not obvious. However with an assumption of repetitivity of $\mathbf{\Lambda}$, we get a notion of multi-tilings which is introduced in [19]. Can Theorem 5.3 be extended when $\mathbf{\Lambda}$ is assumed to be only repetitive?

In lattice substitution Delone multi-colour sets, modular coincidence was introduced as a condition equivalent to pure point diffractivity, and it was proved to be computable (see [20,23]). Is there an algorithm for checking algebraic coincidence in substitution Delone multi-colour set?

There is a considerable amount of ongoing work on Pisot-type substitution sequences for the study of number theory, discrete geometry, geometrical combinatorics, mathematical quasicrystals and spectral theory. As a special case of substitutions there are Pisot substitutions in one dimension each of whose substitution matrices has one eigenvalue strictly bigger than 1 and other eigenvalues strictly between 0 and 1 in modulus. It has been conjectured that every Pisot substitution dynamical system in one dimension has pure point spectrum. Here the algebraic coincidence is an alternative way to determine pure point spectrum in the Pisot substitutions. Throughout correspondence with Valerie Berthe, it is noted that the algebraic coincidence is necessary if an exclusive inner point exists, which is conjectured to hold for every Pisot unit substitution in one dimension (see [1]). Does every Pisot unit substitution admit algebraic coincidence? Bernd Sing's thesis [30] deals with this problem and provides many equivalence properties to the algebraic coincidence.

Although we have shown in this paper the equivalence between the notions of inter-model sets and pure point spectrum in substitution Delone multi-colour sets, we do not know the measure of the boundary of the window of the inter-model set. When the underlying structure of substitution Delone multi-colour set is on a lattice, we know that the measure of the boundary is zero from [20,23]. It is a remaining question that in substitution Delone multi-colour sets (not assumed to be on lattices) or in Delone multi-colour sets if there is any inter-model set with pure point spectrum whose window has boundary of non-zero measure.

In [3] regular model sets are characterized in terms of their associated dynamical systems. Can inter-model sets be characterized in terms of their associated dynamical systems as well? Would it be possible to extend the equivalence of inter-model sets and pure point spectra for general model sets (not assumed to be substitution Delone sets)?

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